Algebra Based Physics

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Welcome to physics! Physics is the study of many inter-related concepts in nature. When we combine numbers/mathematics with concepts and measuring unit, all sorts of incredible things happen. We will see many magical things but we must first begin at the bottom and talk about units.

Units can be abstractly thought of as concepts (such as space and time) multiplied and divided by each other. A concept is hard to define but easy to teach. Common concepts we use are mass, energy, force, space, time, space-time, etc. The concepts we generally use are irreducible (insofar as we know!) such as mass, distance, and time and from the irreducible concepts we can construct many many new concepts.

Units are the words we use for the marks and ticks on our clocks, scales and rulers.

The unit of space we will commonly use is the meter. In the USA we tend to use a king's foot as our unit of space. Since they both describe the same concept (distance) they should be related somehow.

E.g. if your ruler says 1 meter a mine says 1 foot then my ruler will be about 1/3 as long as yours. This is called a conversion factor.

$$1 \text{ foot } = 0.3048 \text{ meters}$$

We get the conversion factor by dividing

$$\frac{1\text{ft}}{0.3048 \text{ m}} = 3.28\frac{\text{ft}}{\text{m}}$$

If we have a distance written in meters then we can convert that into feet by multiplying our measurement by the appropriate conversion factors. For example,

$$7m \xrightarrow{\text{into ft}} 7 \times 3.28m\frac{ft}{m} = 22.96ft$$

The key to notice is how the "units" cancel. We have meter in the numerator and meter in the denominator. We imagine "cancelling" the units just as though they were algebraic numbers (BUT THEY'RE NOT!!). What is left is "feet" in the numerator. Schematically meter. This is a super power of physics which mathematics has no access to. Legitimate equations in physics must have proper units. For example, $x = t^3$ is assuredly NOT a legitimate physical equation. On one side we have distance and on the other side we have time cubed - viz, $m = s^3$ but meters equals meters! Another example might be $x^2 = d + t - \cos(E)$ where x is a distance, d is a distance, t is a time, and E is an energy. There is absolutely no sense in which we can "add" a distance and a time and I cant even begin to imagine what a unit of $\cos(energy)$ could possibly be!

Now, to solve a problem we may need several conversion factors in a single computation

$$5 \text{ miles} \times \left(5280 \frac{\text{ft}}{\text{mile}}\right) \times \left(\frac{1 \text{ m}}{3.3\text{ft}}\right) \times \left(\frac{1\text{km}}{1000 \text{ m}}\right) = 8 \text{ km}$$

If you keep track of the numerator and denominator and cancel units accordingly then you can convert just about anything! Advice: always think about your units.

Significant Figures, Accuracy, Precision.

Physics is ultimately an experimental science. We never obtain exact results in the lab so we must compare our limited measurements with our physical model or theoretical predictions. This is one of the single most important concepts to keep in mind when studying (verifiable) reality - experimental results are different from theoretical predictions. The scientific method in physics is the ballet between the two sides of the scientific coin.

For example, it could happen that our theory predicts a mass to be

$$m = 5.0000372 \text{ kg}$$

but in the lab the best we can measure mass with our instruments and scales might be around

$$m = 5.00 \text{ kg} \pm 0.007$$

The $\pm$ sign is due to sometimes measuring a little more and sometimes a little less. We cannot be "certain" of a particular mass within this range. It could be because we, as humans, make a mistake, or it could be a systematic "error" in the preparation or conduction of our experiment. Either way, we make several independent measurements of the same experiment and average our results. The $\pm$ is called the "uncertainty" or "variance/standard deviation" of our experiments. We ALWAYS include our "error bars". If you come across a scientists without error bars and uncertainties you should be VERY cautious about integrating their results into your world view of reality! Charlatans are certain; philosophers and scientists are uncertain.

Now, perhaps some bright student one day finds a more accurate method/experiment to measure mass and they find

$$m = 5.0000150 \text{ kg} \pm 0.00000004$$

In collaboration with the student we hurriedly send off a manuscript for publication claiming that nature does not agree with the currently accepted model of physics (we get two distinctly different results)

The theorists then all get together and propose and publish several new models or theories that agree with our precise" measurement of massFurthermorewe know it is precise because several other researchers have all produced their own independent measurements very close to ours using a variety of independent methods

Accuracy is how close a measurement is to the correctvalue But who is correct! Only nature can tell us Generally it depends on how many decimal places you can achieve

Our most accurate measurement is the ratio between inertial and gravitational mass by the MICROSCOPE experiment in 2017 which found

$$\frac{m_g}{m_1} = 1.00000000000000(99)$$

Whether this is precise remains to be known - we only have a single experiment and that experiment may later be found to disagree with other independent experiments. It could be that nature does have a $m_g/m_i > 1$ but if so then it would need to be less than or equal to this experimental value if our measurement is correct! In physics we generally believe this ration must be 1 exactly. This however, is an assumption we impose, and every time we've asked nature, she's responded with "if it's not 1 then it's pretty darn close". A value not equal to unity would be a MAJOR deal in our understanding of the universe.

That's our most accurate physical result. Our most "precise" measurement goes to the magnetic dipole moment of the electron.

$$\frac{g}{2} = 1.00115965218085(76)$$

Several independent experiments have found this same result and is in excellent agreement with theory!

"Precision" refers to how close measurements are to each other.

The goal of experimental science is to obtain both accurate and precise measurements, These in turn allow us to develop new theories and hypotheses which then allow us to create new predictions thereby completing the scientific feedback loop - we slowly, inexorably, uncover verifiable reality one experiment at a time.

Uncertainty and Error

All measurements fundamentally have limitations, or, said another way, each measurement only yields a finite amount of information.

Uncertainty allows us to quantify the difference between what we measure and what we expect. In general, uncertainty comes in two classes: systematic and statistical. Systematic uncertainty is due to the resolution of our measuring equipment and statistical is due to the inherent randomness of the universe and our lack of total information. These are usually written $\delta S(stat, sys)$: δ is the Greek letters delta and here tends to mean "small difference".

Often we calculate percent uncertainty between theory and experiment

where S is our expected result. For $\frac{m_g}{m_i}$ we expect 1 So,

%Uncertainty $= 10^{-15} \times 100 = 10^{-13}\%$!

A tiny number indeed!

Significant figures

When manipulating measurements algebraically we must be VERY careful not to create or destroy any known or discovered information.

For example, suppose we measure $m = 1.00$ kg but for some reason we need to divide by 3 in order to calculate some other quantity for our study.

Well $\frac{1.00}{3} = 0.3333\ldots$

The problem is we absolutely do not know this quantity to infinite precision! It is as if we summoned information from the bowels of mathematics and planted it into our physical universe!

If we are careful scientists then we'd better keep only a finite number of decimal places.

But how many?! Well, there are many rules (see handout). In this particular case we would keep only three of the 3's!

$$\frac{1.00}{3} = 0.333 \text{ kg}$$

The numbers we have kept are called significant. Sometimes, in some cases, a particular number is not significant.

1.00 is said to have three "sig figs" whereas 0.33 is said to have just two. We see that we must be careful about how many numbers we keep in a computation however long it may be. The leading zero 0.33 is considered insignificant. Be sure to pay close attention to the rules!

Vectors

Now we arrive at the mathematical pre-requisites we shall need to study physics. We use many many mathematical tools to study nature and verifiable reality but perhaps the single most useful and important mathematical tool is what's known as a "vector".

In physics we say "a vector is anything that if you rotate it by 360^o it returns to exactly where it started". You might wonder if there are even things that don't behave that way and indeed there are!

Vectors have a very particular mathematical definition but here we shall only consider that vectors are mathematical objects described by two real numbers. One of the numbers is magnitude or length and the other is called direction (angle).

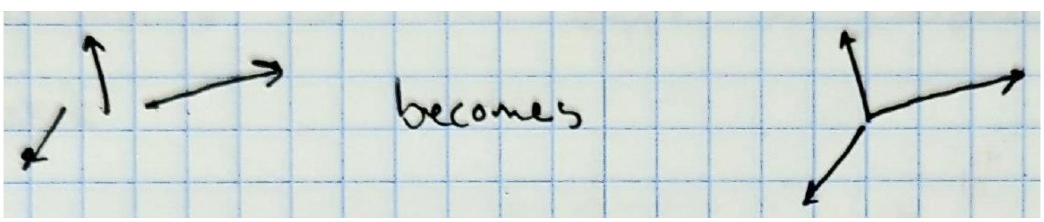
Scalars are objects described only by a single real number. We can also say "a scalar is a quantity that doesn't change at all under rotation".

Geometrically, we can imagine vectors as arrows lying on a two dimensional table. We can have a whole "field" of vectors all scattered about.

An important rule we will repeatedly leverage to our advantage is that we are free to move the arrows ANYWHERE we'd like so long as:

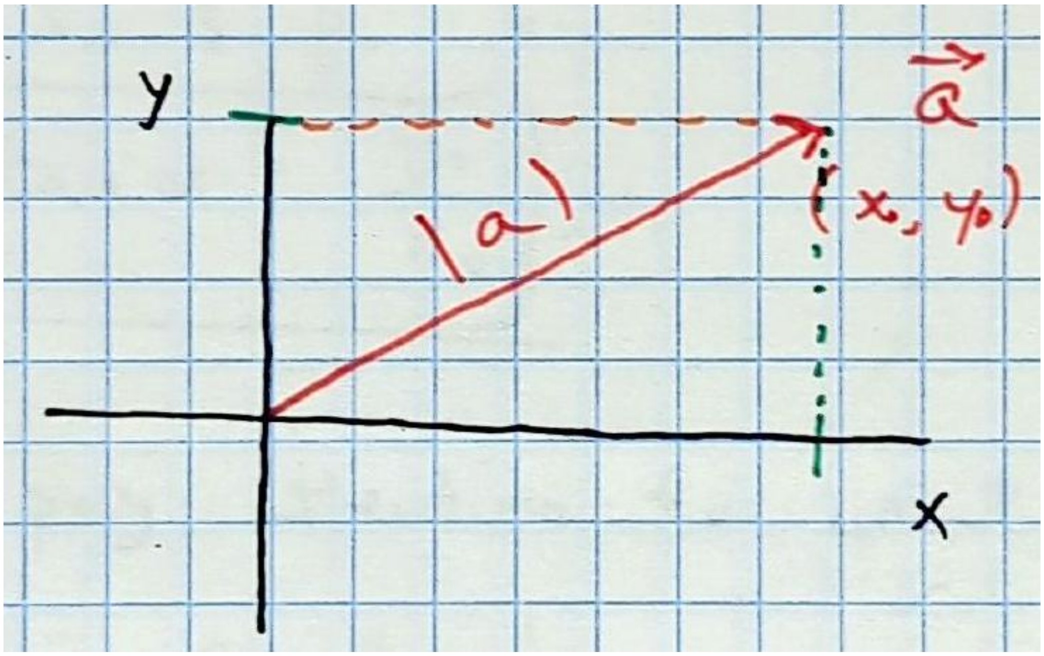
1. you do NOT stretch or shrink the arrow
2. you do NOT change its direction

Apart from those two rules it's a free country! For example, below we happen to decide to put all the vectors at a single point



Incredible things happen when we pick up our vectors and drop them into an (x, y) plane! We will call this "choosing coordinates". You can choose ANY coordinate system you'd like.

Let's call some particular vector $\vec{a}$ with a little arrow on top to remind us it is not a number, or a function, or anything else: it's a vector. Below we have chosen coordinates (x, y) and placed the "tail" of our vector at the origin. See below



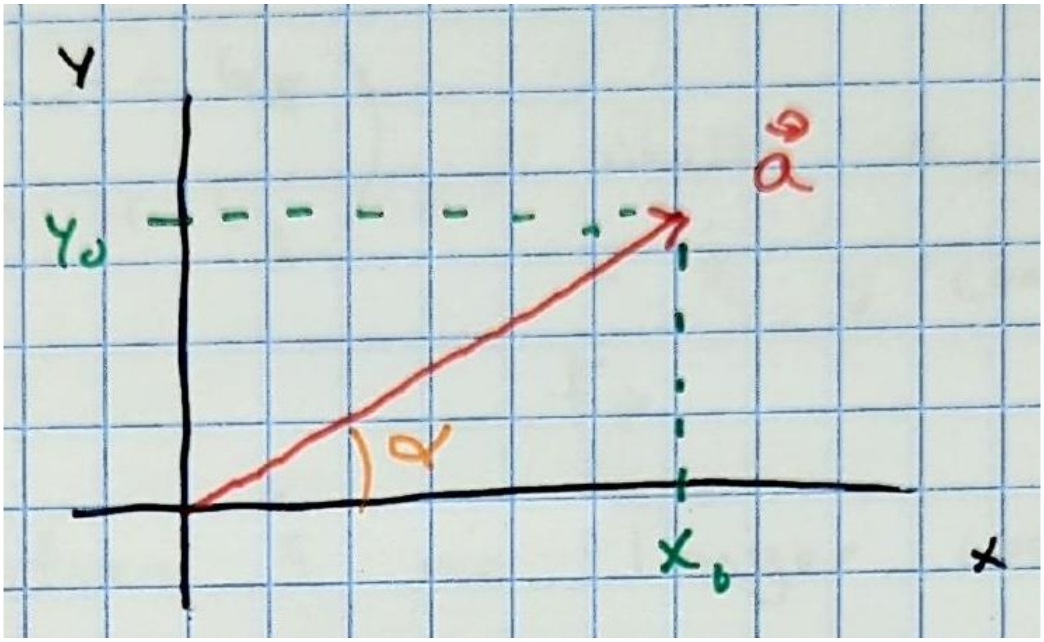
If the tail is at $(0, 0)$ then the tip will be at some particular point in the plane: call it (a_x, a_y) - it's just a name for two real numbers.

In this case a_y forms the vertical edge of a triangle and a_x is the base. Now we can ask Pythagoras for the value of $|\vec{a}|$ called "the magnitude" or "length" of $\vec{a}$

$$|a| = \sqrt{a_x^2 + a_y^2}$$

This is one of the two numbers we need to describe our vector.

We may also ask for the direction. According to Pythagoras this would be the angle α (alpha) measured from $+x$ axis as below. (This is the definition of what we mean by direction. Always the $+x$ axis!)



Recalling trigonometry the direction is related to a_x and a_y by

$$\tan\alpha = \frac{a_y}{a_x}$$

Be sure you pay attention to what quadrant you're in when using this equation!

Now, sometimes we are given a_x and a_y but often we are given a priori $|a|$ and α. We will see later that most of the time we wish to describe vectors in terms of (a_x, a_y) rather than the length and direction. So we must develop a dictionary to translate one version of a vector into the other.

The pair (a_x, a_y) is called the "Cartesian Coordinates" of $\vec{a}$ and the length and angular direction is called the "polar" form of $\vec{a}$. If we are given the polar form then we can get the cartesian form by the trigonometric relations

$$a_x = |a|\cos\alpha$$
$$a_y = |a|\sin\alpha$$

We often write vectors as columns of the Cartesian coordinates.

$$\vec{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

This is called a "column vector".

Now, just like numbers (what we now call "scalars") we can define an algebra of vectors. That is to say, we shall come up with a notion of "adding, subtracting, and multiplying" vectors (perhaps it's curious I don't mention dividing!) Now, vectors are assuredly NOT numbers; they're more abstract. They're kind of a data structure and because of this there may very well not be a sense in which we may compose two vectors into something that looks like our everyday addition or multiplication. For this reason, we shall tentatively write $\vec{a} \oplus \vec{b}$ rather than $\vec{a} + \vec{b}$ since the "addition" of vectors might have nothing in common with that of the real numbers we're so used to!

So let me tell you have adding vectors works and I will let you attempt to prove it.

Suppose we have two vectors $\vec{a}$ and $\vec{b}$. Let us choose coordinates and write these two vectors in their Cartesian form. Then we have

$$\vec{a} \oplus \vec{b} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \end{bmatrix} = \vec{a} + \vec{b}$$

This reads as "to sum two vectors you add their x and y components together in kind". Since this looks a LOT like our typical real number addition we will no longer write $\oplus$ and simply write our usual $+$ sign.

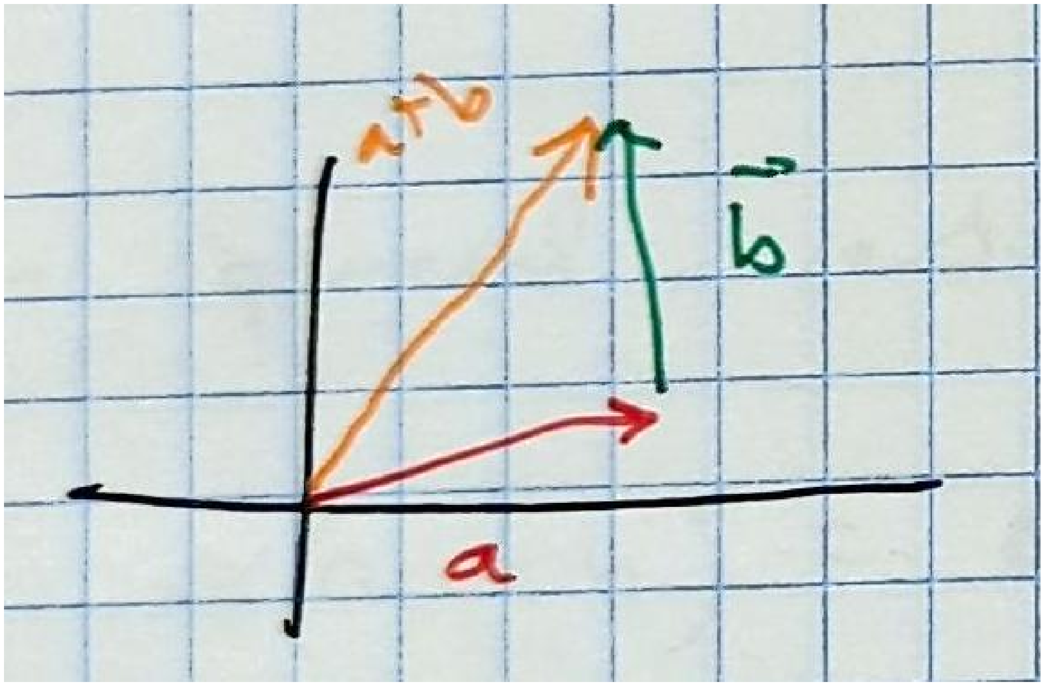
Using this show that $\vec{a} + \vec{b}$ is NOT given by $(|a| + |b|)$ and $(\alpha + \beta)$. We never add vectors with their polar forms. The addition of two vectors in polar form is incredibly convoluted and clumsy - it can be done, but it's a MESS. Always decompose your vectors in cartesian coordinates and THEN add them.

$\vec{a} + \vec{b} = \binom{a_x + b_x}{a_y + b_y}$ is called the "analytic" method of vector addition.

We also have ways to "eyeball" the sum of vectors geometrically. If we want $\vec{a} + \vec{b}$ then this will be some NEW vector we call $\vec{c}$.

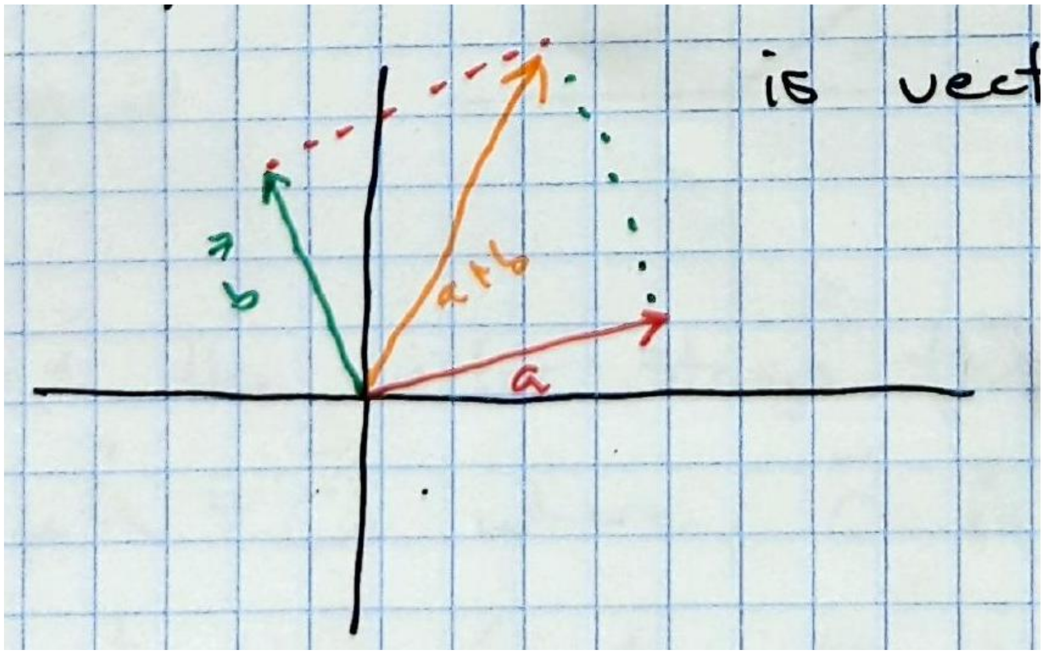
Geometrically, we have two routes: "tip to tail" and "tail to tail".

"tip to tail:" take vector $\vec{b}$ and drag its tail to the tip of $\vec{a}$. Then draw an arrow from the tail of $\vec{a}$ to the tip of $\vec{b}$ as below. This new vector is the sum of the two.



tail

tail to tail: also known as quadrilateral method. Place both vector's tails at the origin then create a quadrilateral. The line bisecting this quadrilateral from the tails is the sum of $\vec{a}$ and $\vec{b}$



y

Try proving these methods from $\vec{a}+\vec{b}=\binom{a_x+b_x}{a_y+b_y}$

There are more than a few types of multiplication we can define for vectors but we will NOT multiply vectors together just yet.

First we shall show that we can multiply scalars and vectors together. Suppose $\mu \in \mathbb{R}$ is a real number valued scalar. Then,

$$\mu\vec{a} = \begin{bmatrix} \mu a_x \\ \mu a_y \end{bmatrix}$$

For example,

$$5\vec{a} = \begin{bmatrix} 5a_x \\ 5a_y \end{bmatrix}$$

Easy!

In general vectors are linear

$$\alpha(\vec{a}+\vec{b}) = \alpha\vec{a} + \alpha\vec{b}$$

where $\alpha \in \mathbb{R}$ is a scalar.

This scalar multiplication stretches or shrinks the original vector. Show that the direction stays the same but the magnitude is multiplied by the scalar.

If $\mu = 1$ the vector stays the same and if $\mu = 0$ then $\mu\vec{a} = \vec{0}$ where $\vec{0}$ is the "zero vector". It works just like 0 the number.

For now, thats all the tools we will need to study physics.

Kinematics

Displacement, Velocity, Acceleration

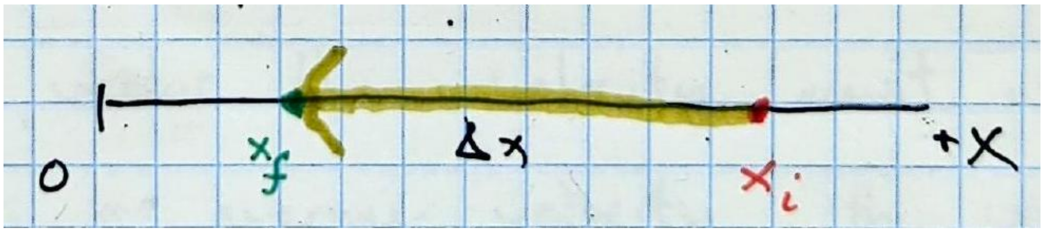
In physics concepts are most important. New concepts hove caused Kuhnian paradigm shifts from Newton to modern times. We may in fact be on the cusp of new concepts in physics with the help of artificial intelligence and human creativity. Some of the most fundamental concepts in physics are displacement, velocity, acceleration, and time. This is the study of "Kinematics" and we will make frequent reference to the great philosopher and scientist Galileo who performed many experiments to determine the relationships between these concepts.

Displacement is the vector change in position of an object. It can point in any direction and have any length. In one dimension we write

$$\Delta\vec{x} = \vec{x}_{final} - \vec{x}_{initial} = \vec{x}_f - \vec{x}_i = \vec{x}_f - \vec{x}_0$$

Each notation is apt. Importantly it is always FINAL minus INITIAL. The opposite order will produce a minus sign (180 degree change in direction) so be extra careful.

Below $|\vec{x}_f| < |\vec{x}_i|$ and so $\Delta\vec{x}$ points in the $-x$ direction the tip of the vector Δx is always at x_f



Now, given a displacement $\Delta\vec{x}$ we can define distance traveled by its magnitude. Let us call this distance d, then

$$d = |\Delta\vec{x}|$$

d is strictly greater than or equal to zero. d can NEVER be less than 0 - that is what "magnitude" or "length" means! the displacement itself can be in some "minus" direction but its length is always positive.

The next concept we define is time. The elapsed time of a process is

$$\Delta t = t_f - t_i$$

We take time to always move forward and therefore $\Delta t \geq 0$. t_f and t_i are the readings on our clocks at the final is initial points during some physical process. Note further that this is assuredly NOT a vector. Time is one dimensional and will always proceed forward in time.

Next, we may take various combinations of this displacement and duration in time. Velocity is the amount of displacement per some unit of time.

$$\vec{v} = \vec{v}_{\text{avg}} = \frac{\Delta\vec{x}}{\Delta t} = \frac{\vec{x}_f - \vec{x}_i}{t_f - t_i}$$

Since $\Delta\vec{x}$ is a vector then velocity must also be a vector. Importantly $\vec{v}_{\text{avg}}$ is an average velocity; the velocity over a nonzero amount of time. We can define an "instantaneous velocity" if $\Delta t \to 0$ but this would take us into calculus to pursue properly. For our purposes the average velocity will be taken to be synonymous with "the velocity".

Since velocity is a vector then we can call $|\hat{v}|$ the speed s where $s \geq 0$. It is crucial you dont get "speed" and "velocity" mixed up. We must abandon colloquial definitions and stick to firm definitions of these ideas!

Next we could define the average acceleration as the change in velocity over a time interval. This will manifestly be a vector quantity

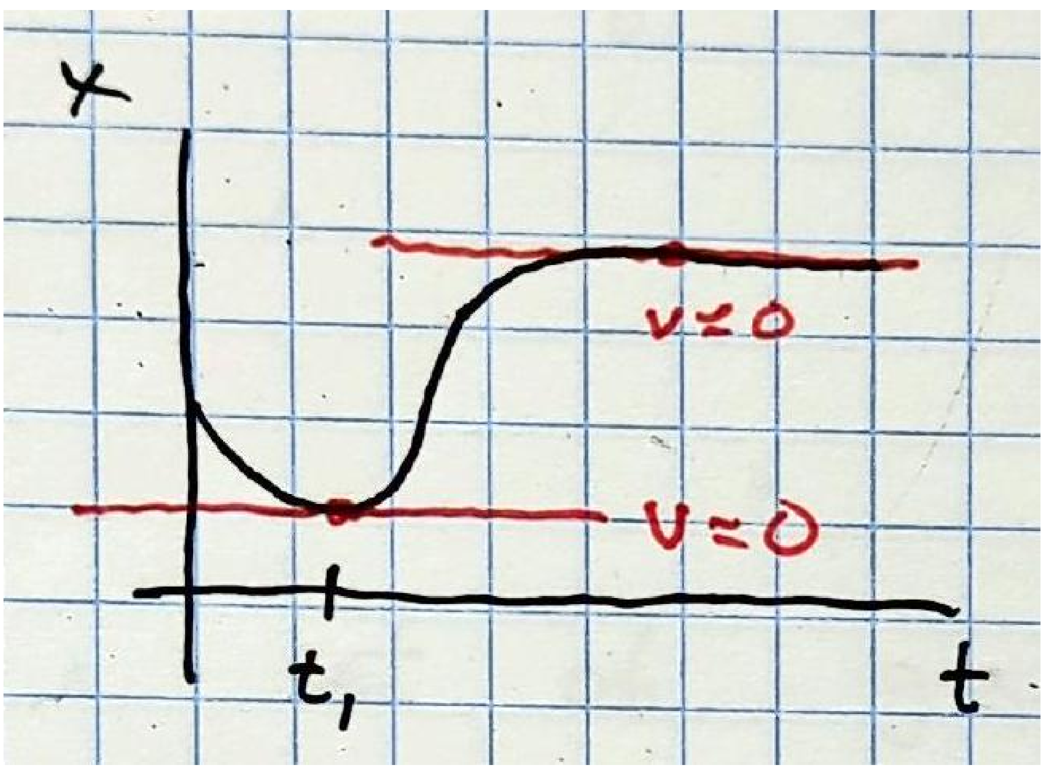
$$\vec{a} = \vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

This is how an object's velocity changes over a time interval. Near the surface of the earth an object dropped will accelerate downwards towards the surface of the earth at $-9.8m/s^2$. it seems like a funny set of units but this simply means that every second that passes adds another $9.8m/s$ to the objects speed. It is speeding up. If it were slowing down we would say it is "decelerating".

There are deep inter-dependent relationships between these kinematic variables. For example, if you have a plot representing the position as a function of time then one can show that the slope of the tangent line to the plot at some specific time is the instantaneous velocity. Similarly, the slope of the velocity curve as a function of time is given by the acceleration.

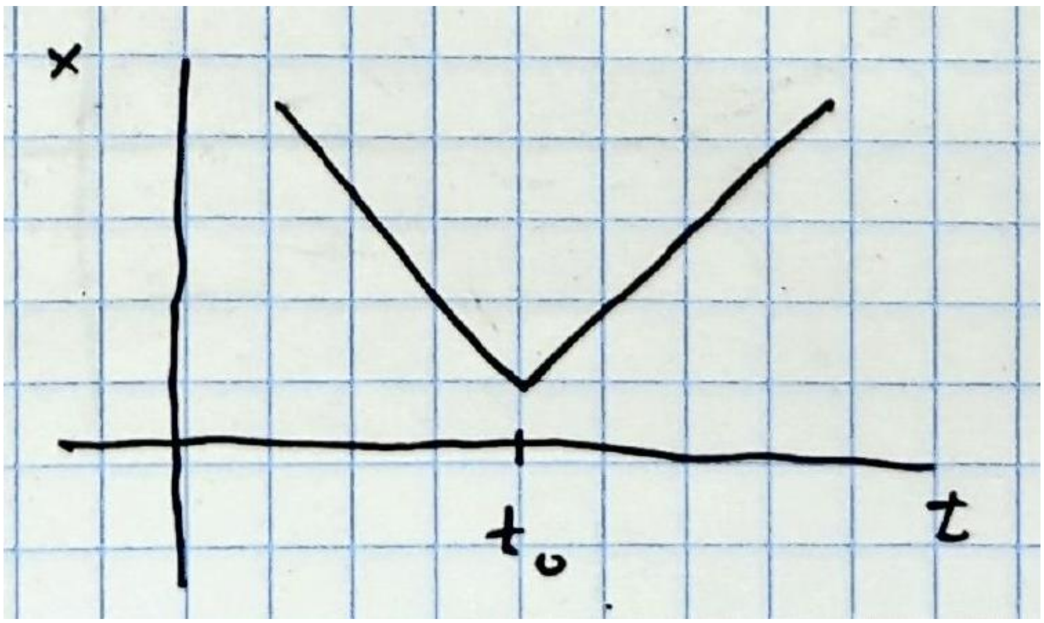
In general you might be given a graph and asked to find certain special values of such a plot.

For example,

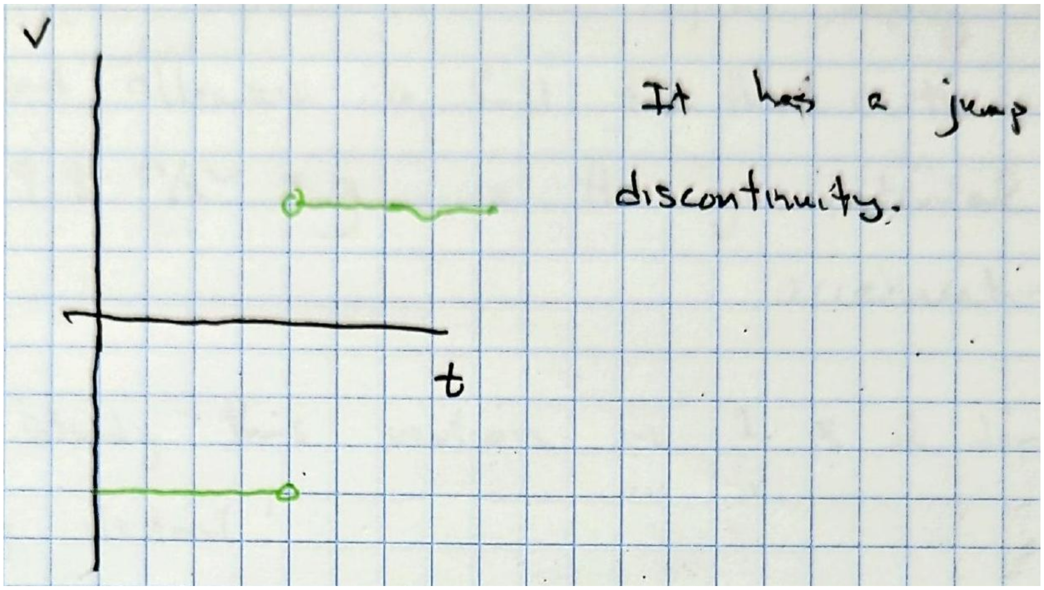


We can find where the velocity is zero by the slope of the tangent line at a point. In this graph the object decreases its position and right at t_1 it begins to turn around.

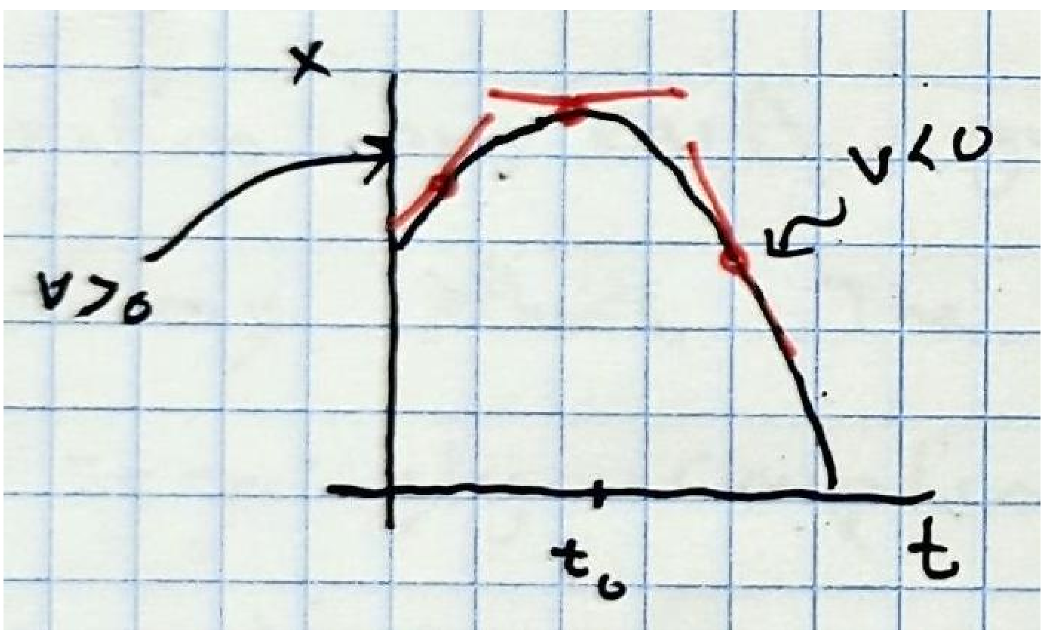
In order to change directions from positive to negative velocity one must go through zero. The exception is where something is suddenly kicked ito a new direction. That would look like the plot below



Here, the slope at time $t = t_0$ is said to be undefined since the velocity graph would have a jump discontinuity at t_0.

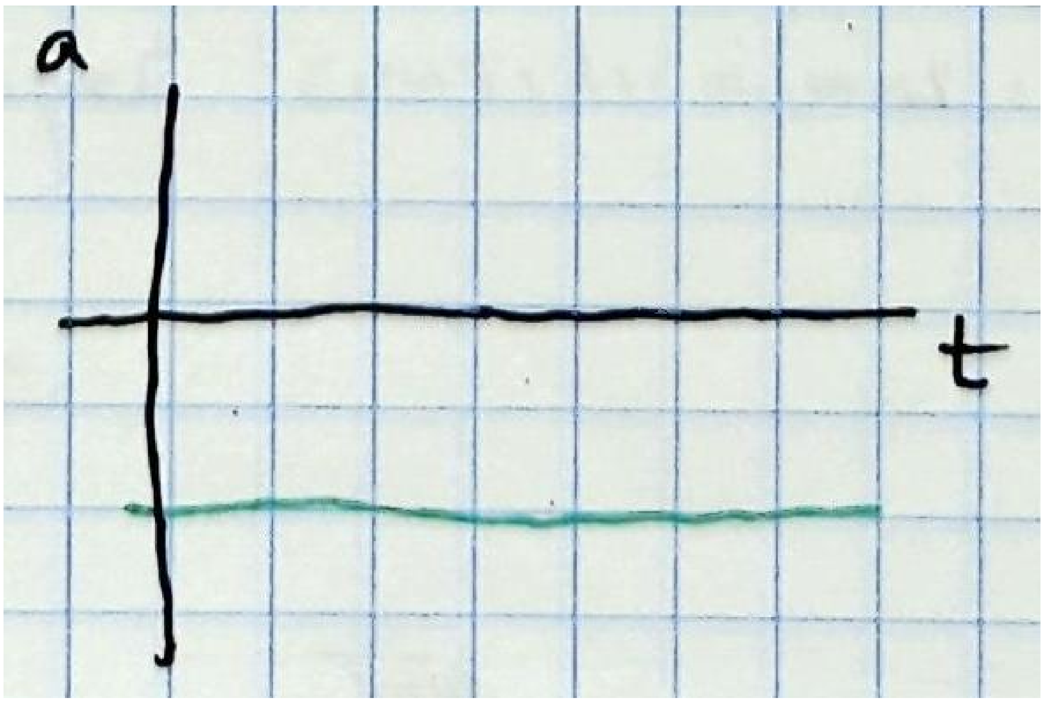


Let's look at a set of graphs:



From the $x(t)$ graph we can sketch $v(t)$ to find a straight line (if $x(t)$ is parabolic) with negative slope which crosses the time axis at $t = t_0$.

Similarly the slope of $v(t)$ yields the acceleration $a(t)$ which in this case is a straight line at $a < 0$. (see below) $a(t)$ in this particular case is constant at all times.



We will become very familiar with this type of motion. This graph represents an object being thrown upwards and allowed to fall back to the earth. $a = -9.8m/s^2 \equiv g$ is called the "gravitational acceleration". Every object near the surface of earth experiences this constant acceleration downwardly oriented.

We will study this motion in both 1 and 2 dimensions in gory detail!

Galileo

We have a continuous theoretical model of motion near the Earth which was given to us by Galileo and Newton (among others). The model is given by some seemingly complicated equations but these equations are surprisingly accurate and simple (as you will soon discover, with lots of practice)! Galileo found these expressions them "phenomenologically" or "empirically" meaning that he did experiments and noticed over time that the data fit nicely to these specific graphs. Newton, 100 years later, went further and "derived" them from more general and deeper considerations.

Galileo's kinematic equations for constant acceleration are;

$$\begin{aligned}
&x_f = x_i + v_i t + \frac{1}{2}at^2 \\
&v_f = v_i + at \\
&v_f^2 = v_i^2 + 2a\left(x_f - x_i\right)
\end{aligned}$$

This is a LOT to unpack!
First, the subscripts are nothing more than labels - a book keeping device, where

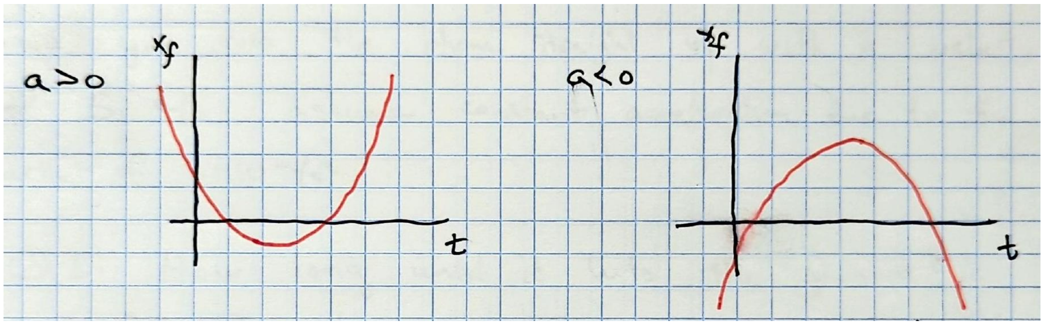
$$\begin{aligned}
f &:= \text{ final} \\
i &:= \text{ initial}
\end{aligned}$$

so v_f mean "final velocity". In one dimension of space it is a $\pm$ number that represents a vector's length and direction (negative direction and positive direction). All motion in this 1D case takes place on a line whether vertical, horizontal, or some other line.

Secondly, despite all the crazy letters every where we only consider t as a "variable". If we "plugin" $t = 5\text{s}$ then these equations will give us a number we call x_f or v_f which tell us the position and velocity at $t = 5\text{s}$.

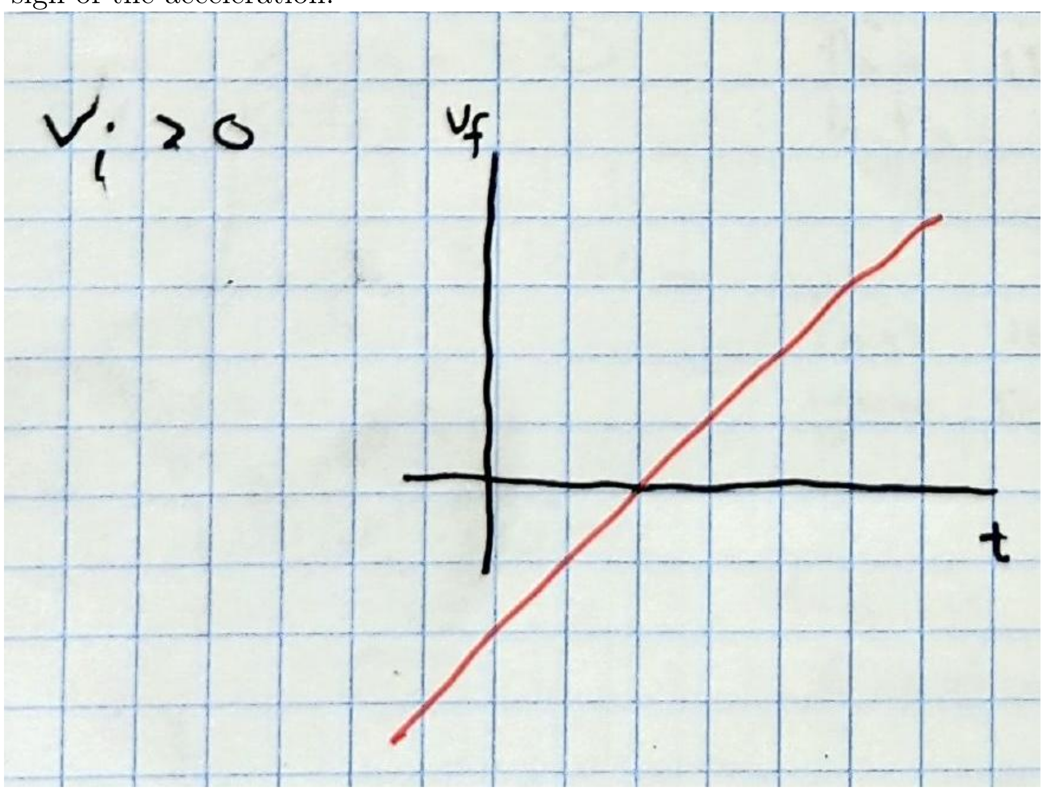
Perhaps you noticed that if t is the variable and $x_f(t)$ is the value (everything here are numbers) then this looks just like the familiar parabola you've studied so well.

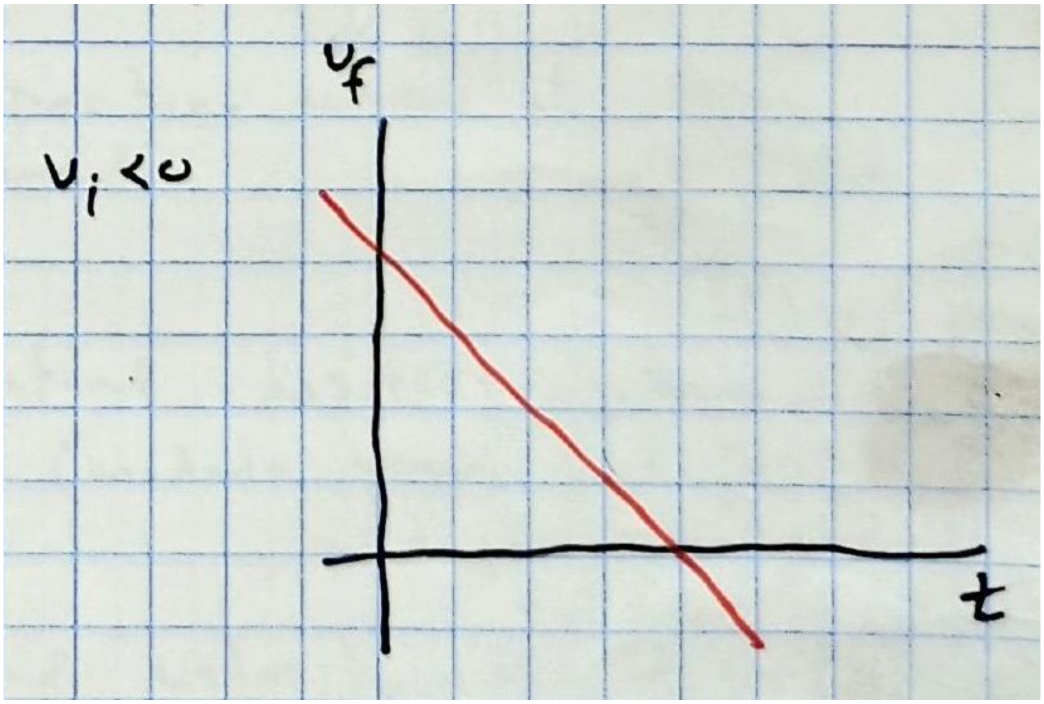
Consider $a > 0$ and $a < 0$. if the acceleration is positive then our parabola opens up and if the acceleration is negative opens down.



Nevermind that I allowed for negative values of t here. At this level we are only considering the mathematical functions.

Similarly $v_f(t)$ looks like a straight line sloping up or down depending on the sign of the acceleration.





Let's see what these equations can do!

Example of Vertical Motion Near the Earth

Suppose you climb Mt. Baldy at Garner State Park and decide you no longer want your phone. Let's say that the top of the cliff is about 1000 meters (it's more like 1000 feet) and you toss the phone straight up with a velocity of 5m/s. Assume constant acceleration due to the Earth of magnitude $10m/s^2$ downwards.

Q1: How long until it hits the ground?

The hardest part to physics involves translating words and ideas into firm mathematical statements which we can manipulate algebraically.

We will typically follow a strict methodology I outline below as follows:

1. Draw it!

2. Choose your favorite coordinates (rulers and clocks)

3. Write everything you know using these coordinates.

4. Phone a friend

5. fill in the relevant equations given to you by your friend.

That's it! If you make a habit of following these 5 steps then you'll likely be able to solve your physics problem.

The first step is crucial - draw what's happening. After decades studying physics I still draw pictures. Visualizing your situation is probably the most important step where you galvanize your intuition.

Our second step is vastly under-appreciated by high school teachers everywhere. We demand that the things YOU observe with your clocks and rulers should be the exact same as what I observe with MY clocks and rulers. We may even use different units - the physics shouldnt matter.

When we choose our coordinates we are picking a special place with which to measure everything from. We might place our rulers at our feet and set our stopwatches to $t = 0$ seconds or i might rather prefer to put my rulers up on top of a cliff and watch the physics unfold with binoculars. It is YOUR choice and some of the most beautiful results and methods in all of physics revolve around the idea that the universe doesnt care where you put your coordinates - here, Mars, andromeda, etc. Just don't try to put your coordinates in a blackhole!!

Third step: once you have picked out your coordinates only then do you start filling in your data. For example, if I put my origin at the top of a 1000 meter cliff and my object is initially at the same point up there I would fill in $x_i = 0$ but if my coordinates were placed at the bottom of the cliff then the initial object way up on top of the cliff would be located at $x_i = 1000$ meters. It is yet clear that regardless we wind up with the same results but I contend they must produce identical physics despite the numbers being so darned different!

Fourth step: phone a friend. Here is the moment that you go around asking what all your friends have to say. In this course you will make good friends with Galileo, Newton, Joule, and others and being such busy people they tend to only yell into the phone with an equation or two. Here we've only met one person so far, Galileo, who kindly reminds us of the two equations he worked so hard to produce. As time goes on he'll get increasingly fussy at us for bugging him so much - it may help to write them down or memorize them!

Finally, once you've asked your friends for their equations you can begin filling them in with everything you know about your problem and then attempt to see if youve any hope at all of figuring something out from them.

That's your 5 step method. lets begin - How long until the phone hits the ground?

Step 1: Draw it.

Step 2: Choose Coordinates

I wish to place my coordinates such that the origin is at the ground. I could've easily chosen them to be at the top of the mountain but I prefer the ground.

Step 3: write known quantities:

Well, I expect it will hit the ground at the ground so my final position I hope is at

$$x_f = 0$$

Next, I was told the phone starts on top of the 1000 meter mountain so according to my coordinates

$$x_i = 1000$$

Next, I'm told I throw the phone upwards at 5 m/s so id better have

$$v_i = +5$$

Notice that if i threw the phone downwards it would have to be -5 m/s - these are 1D vectors so $\pm$ gives me directions.

Next, Ive been told that the acceleration due to gravity near the earth is $-9.8\ m/s^2$. Since we're friends and not computing anything crucial lets call this

$$a = -10$$

I think that this is all of our data but it's always possible I'm not noticing something. Often in problem solving you think you addressed everything only to realize you've forgotten something important after hours of blood, sweat, and tears. So be it. That's life. Let's see where we get with what we've got.

Step 4: call a friend

We only know one person: Galileo. He assures us that we only ever need the following two equations

$$x_f = x_i + v_i t + \frac{1}{2}at^2$$

$$v_f = v_i + at$$

But he also gave us a third since he's such a nice guy. You can look at it above. I encourage you to get comfortable solving everything with these two primary equations and as you get experienced then you can use the 3rd to save some time and energy.

Okay, step 5: see if we can fill these in using our coordinates and known quantities.

The x_f equation becomes:

$$x_f = 0 = 1000 + 5t - \frac{10}{2}t^2$$

and the v_f one is

$$v_f = 5 - 10t$$

Now, I don't yet know v_f so i just write v_f. IU also dont know what t is. Everything else i have.

Lets take stock. I have here 2 equations and 2 unknown quantities. HAPPY DAYS! WE CAN SOLVE THIS PAIR OF EQUATIONS!

Always ask yourself "how many equations do I have and how many unknowns". If you have more unknowns than equations then you're sunk and youll need to find more equations you hope to leverage.

Now, the $x_f = 0$ equation by golly sure does look like a quadratic that we could solve for t. We will get two values mathematically. Physically this will give us the amount of time it takes to reach $x_f = 0$, the ground, given that $x_i = 1000$.

Doing so yields two distinct times. One of them is negative and the other is $t_f = 14.6$ s. That's mathematics. The physics/universe says "throw away the negative time! We only care about the future after $t = 0$!"

There we have it: we solved our first physics problem: the time it takes for the phone to hit the ground. It's kind of a long time and we ignored a LOT of physics: wind resistance, size/shape of the phone, cross wind, and on and on. Our experiment will assuredly not match $t = 14.6$s but it should be, and will be, pretty stinkin' close!

Notice, we could also plug in this time into the velocity equation to find $v_f = -141$ m/s in case we wanted to know. That's pretty quick! Don't let it hit your head!

Q2: You decide to save your phone. How long do you have before you can catch and save it?

Okay, unfortunately with physics questions we have to make some assumptions when converting colloquial words into strategy. Here Let's suppose "catch" means that it goes up and comes down and we are unable to catch it once it gets just an iota below the cliff.

Same methods: draw it, choose coordinates, knowns, friend, plug-in.

We shall use the same coordinates but here we no longer have $x_f = 0$. We don't want it to hit the ground! Do we know where we want it to end up? Sure, anywhere above or at 1000 meters. Let's catch it exactly when it gets back to 1000 meters. Then

$$x_f = 1000$$

But this is also $x_i = 1000$. So be it! Plugging in:

$$1000 = 1000 + 5t - 5t^2$$

or

$$5t = 5t^2 \implies t = 0, 1$$

Physically we choose $t = 1$ as our solution. Of course it is at 1000 meters at $t = 0$! If our equations didn't predict what we already knew then I doubt anyone would give a hoot about Galileo!

We have 1 second to decide whether to catch our phone!

Now you try: How high does the phone go? Hint: it is where the velocity precisely equals zero! Try to solve this last question two different ways!

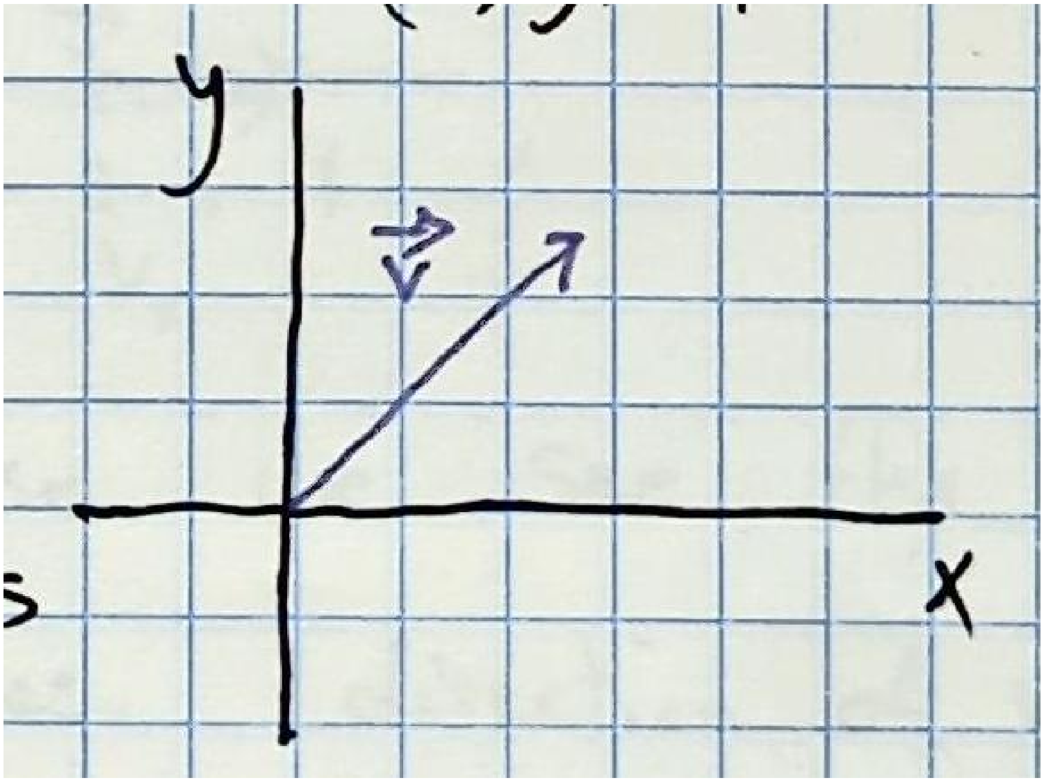
2D Motion

We previously discussed motion in one dimension with constant acceleration. In physics we often study phenomena in 1D and then build up to 2D, 3D, 4D and any dimension (just for fun, i swear!).

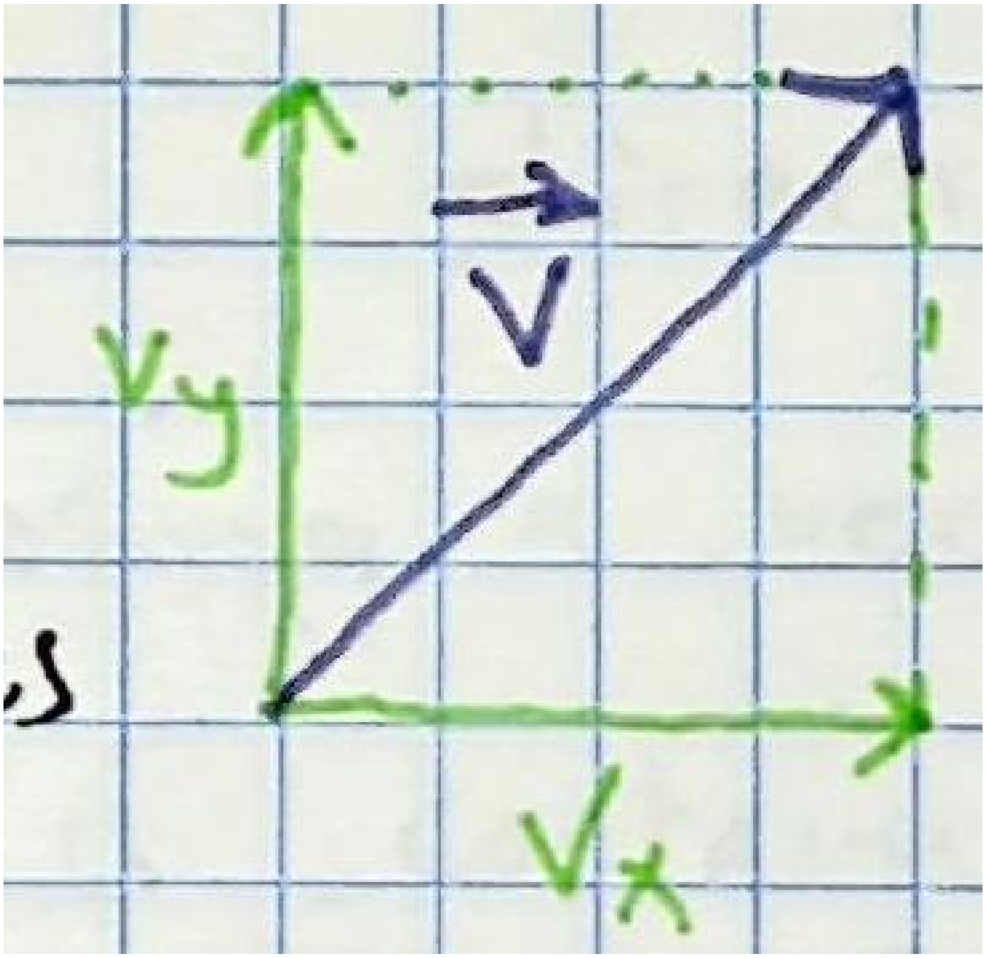
Despite how complex two and three dimensions may seem, it turns out nature has some tricks up her sleeve to allow us to understand higher dimensional motion once we understand 2D motion. After that all of our equations end up generalizing to any dimension!

To get there we must turn back to our 2D vectors we've already forgotten about!

Recall a vector in the (x, y) plane. Let's say it is a velocity vector for definiteness (below).



This vector has components in the x-directions and y direction as follows:



Now, we can consider that v_x and v_y are themselves vectors in their respective single dimensions. They add together, under vector addition, to give you back the total vector $\vec{v}$

$$\vec{v} = \vec{v}_x + \vec{v}_y$$

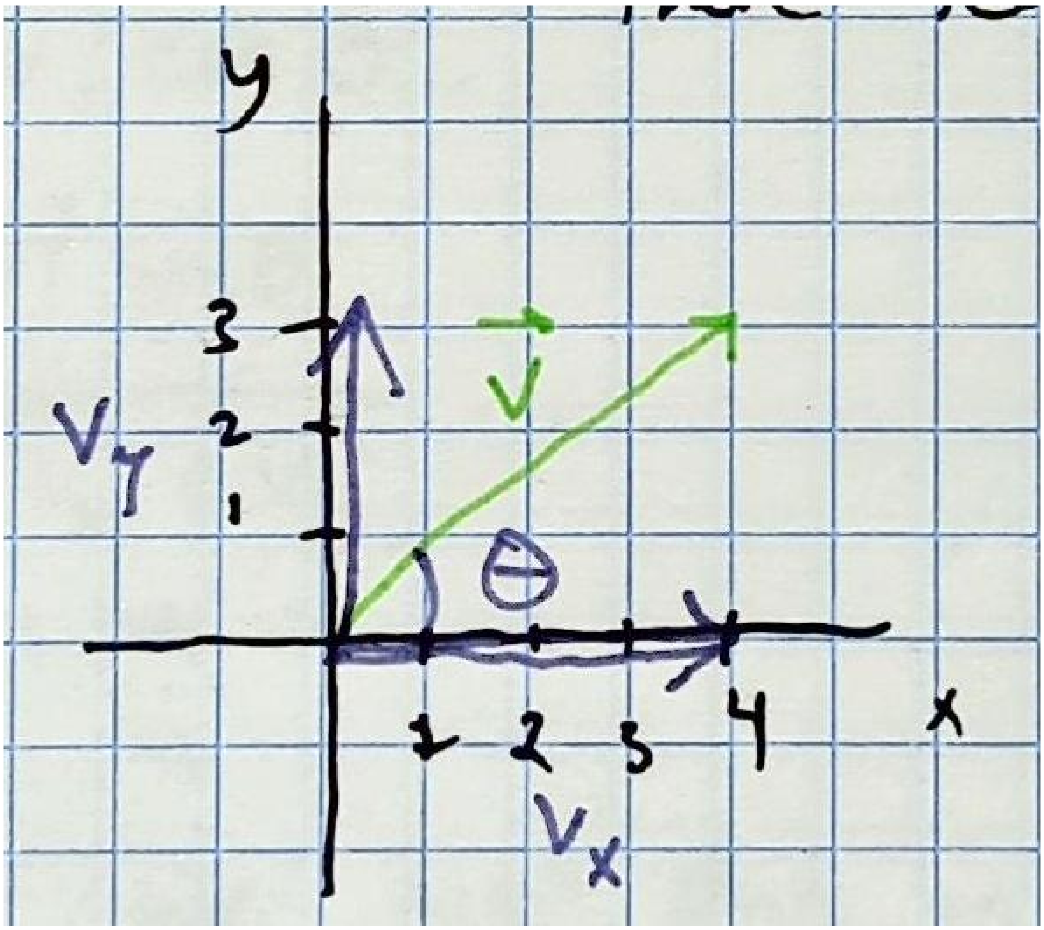
We can put our tools to work is ask what the lengths and directions of these vectors are.

Well $\vec{v}$ makes an angle θ with respect to the x-axis and the magnitude is given by Pythagoras:

$$|v| = \sqrt{v_x^2 + v_y^2}$$

In this equation v_x and v_y are the magnitudes of the component vectors $\vec{v}_x$ and $\vec{v}_y$ but those magnitudes are their lengths along the x and y axes.

In this case (figure below) we can read off what v_x and v_y are



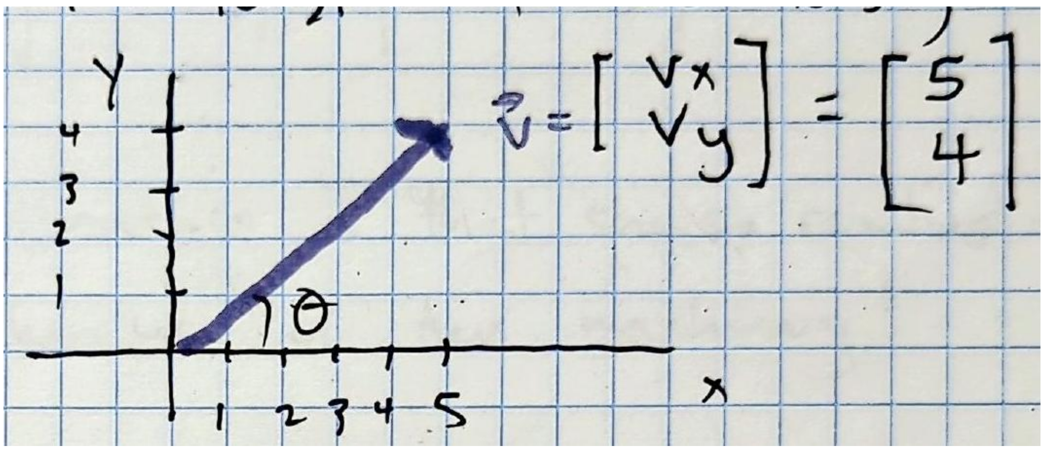
$$|v_x| = 4$$

$$|v_y| = 3$$

This is called "decomposing a vector". The entirety of the method is contained in the two KEY formulas:

$$v_x = |\vec{v}| \cos\theta$$
$$v_y = |\vec{v}| \sin\theta$$

Now, given a choice of coordinates we will almost always take the polar information of a vector and then convert it into Cartesian components as follows in the example below. See the figure:



Recall, our polar to Cartesian (and vice versa) translation dictionary we developed using trigonometry:

$$v_x = |v| \cos\theta$$
$$v_y = |v| \sin\theta$$
$$|v| = \sqrt{v_x^2 + v_y^2}$$
$$\theta = \tan^{-1} \frac{v_y}{v_x}$$

In the specific case above we just plug in our numbers. We know a priori $v_x = 5$ and $v_y = 4$. Therefore

$$5 = |v| \cos\theta$$
$$4 = |v| \sin\theta$$
$$|v| = \sqrt{5^2 + 4^2} = \sqrt{25 + 16} = \sqrt{41}$$
$$\theta = \tan^{-1} \frac{4}{5} = 38.6^o$$

If we wanted to add two or more vectors we can simply add the components as follows

$$\vec{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$
$$\vec{a} + \vec{b} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \end{bmatrix}$$

The beauty of nature is that she so readily allows us to make use of this machinery! Now I make a claim without proof (try to prove it!)

Theorem: Motion in 2 dimensions may be decomposed into two 1 ***Dimensional Motions!***

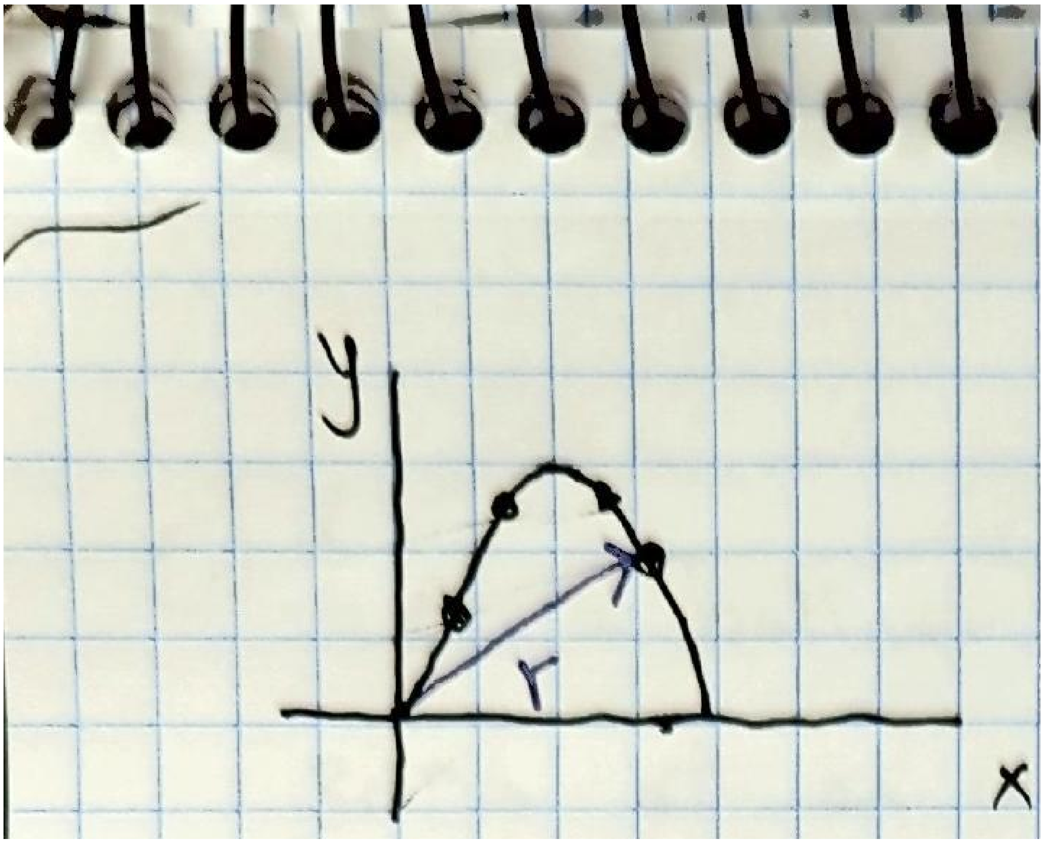
Galileo now tells us the following

$$\vec{x}_f = \vec{x}_i + \vec{v}_i t + \frac{1}{2}\vec{a}t^2$$
$$\vec{v}_f = \vec{v}_i + \vec{a}t$$

Be mindful of All the vectors! This is now a "vector equation". Dear reader, please do not fear this: think of it like a JPEG. It is compressed information.

Not only is the the kinematic equations in two dimensions, this is the equation in ANY dimension! one, two, 348752 dimensions? It is ALL these two equations! JPEG indeed! It's actually lossless compression!

Now, for some coordinate choice $\vec{x}_f(t)$ is a vector that "points" to an object moving along in two or more dimensions. $\vec{x}_i$ points to where the object began at $t = 0$ $\vec{v}_f(t)$ can be thought of as being "attached" to your object and the length and direction is changing as a function of time. $\vec{a}$ in most cases will be a constant vector. In 3D imagine a little fly buzzing around the room such that youre always pointing at it: thats $\vec{x}_f(t)$. see below:



We could write out the x and y directions separately at the expense of twice as many equations

$$\begin{cases} x_f = x_i + v_{ix}t + \frac{1}{2}a_x t^2 \\ v_{fx} = v_{ix} + at \end{cases}$$

and

$$\begin{cases} y_f = y_i + v_{iy}t + \frac{1}{2}at^2 \\ v_{fy} = v_{iy} + a_y t \end{cases}$$

These two sets of equations are connected together by the time variable t.

Example, Suppose we have a ball moving in the x-y plane as in the figure above. Let the position of the ball be described by its $x - y$ coordinates in a vector called "the position vector" (often called $\vec{r}$)

$$\vec{x}_f = \vec{r} = \begin{bmatrix} x_f \\ y_f \end{bmatrix}$$

Suppose the ball was kicked into the air with an initial velocity vector $\vec{v}_i$. This initial velocity has some magnitude and direction. Imagine the path of the ball: it travels along some path that goes up y and along x in some arcing fashion - the above picture is the trajectory in space described by x and y there's no time in the plot.

Now, imagine that the initial velocity points straight up.

then

$$\vec{v}_i = \begin{bmatrix} 0 \\ v_{iy} \end{bmatrix}.$$

In that case we have a 1D motion problem and similarly if

$$\vec{v}_i = \begin{bmatrix} v_{ix} \\ 0 \end{bmatrix}.$$

Then it's purely horizontal motion.

An arbitrary velocity vector will result in a combine motion in the (x, y) plane accordingly. We are almost always given the length and direction of our vectors and then we must decompose them into x, y coordinates using our decomposition routine.

Let's see an example.

Q1: A ball is kicked with initial velocity of $30m/s$ at an inclination of $45°$ to the ground. Where does it land?

Same problem solving steps: draw it, choose coordinates, knowns, friend, plug in.

We drew it, now we choose coordinates. Let's pick our origin to be at the ground precisely where the ball is being kicked from.

Then

$x_i = y_i = 0$ (we choose the origin)
$a_x = 0$ (there is no acceleration along x near the surface of earth)
$a_y = -10$ (downward acceleration due to gravity somehow)

Naturally, we call Galileo who gives us our equations above.

Now, aside from $\vec{v}_i$ we have everything we need. We know the length and direction of $\vec{v}_i$ but we really want the x and y components. We can use our two key equations

$$v_x = |v| \cos\theta = 30 \cos 45 = 15\sqrt{2} \text{ m/s}$$
$$v_y = |v| \sin\theta = 30 \sin 45 = 15\sqrt{2} \text{ m/s}$$

This kind of decomposition is used EVERYWHERE in physics and with practice comes somewhat instantly. We will promote this step to step 3a: decompose all vectors in your coordinates

Now we have

$$\vec{x}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$v_i = \begin{bmatrix} 15\sqrt{2} \\ 15\sqrt{2} \end{bmatrix} = 15\sqrt{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\vec{a} = \begin{bmatrix} 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$$

our equations of motion are:

$$x_f = x_i + v_{ix}t + \frac{1}{2}a_x t^2$$
$$y_f = y_i + v_{iy} + \frac{1}{2}a_y t^2$$
$$v_{fx} = v_{ix} + a_x t$$
$$v_{fy} = v_{iy} + a_y t$$

Or, alternatively, we could, and should write:

$$\vec{x}_f = \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} v_{ix} \\ v_{iy} \end{bmatrix} t + \frac{1}{2} \begin{bmatrix} a_x \\ a_y \end{bmatrix} t^2$$

and

$$\vec{v}_f = \begin{bmatrix} v_{ix} \\ v_{iy} \end{bmatrix} + \begin{bmatrix} a_x \\ a_y \end{bmatrix} t$$

Let's plug in what we know and see where it leads

$$\vec{x}_f = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 15\sqrt{2} \\ 15\sqrt{2} \end{bmatrix} t + \frac{1}{2} \begin{bmatrix} 0 \\ -10 \end{bmatrix} t^2$$

and

$$\vec{v}_f = \begin{bmatrix} 15\sqrt{2} \\ 15\sqrt{2} \end{bmatrix} + \begin{bmatrix} 0 \\ -10 \end{bmatrix} t$$

cleaning up

$$\vec{x}_f = \begin{bmatrix} 15\sqrt{2}t \\ 15\sqrt{2}t \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \end{bmatrix} t^2$$

and

$$\vec{v}_f = \begin{bmatrix} 15\sqrt{2} \\ 15\sqrt{2} \end{bmatrix} - \begin{bmatrix} 0 \\ 10 \end{bmatrix} t$$

Or we can write out each 1D equation separately.
x :

$$x_f = 15\sqrt{2}t \qquad v_{fx} = 15\sqrt{2}$$

y :

$$y_f = 15\sqrt{2}t - 5t^2 \qquad v_{fy} = 15\sqrt{2} - 10t$$

These 4 equations give us everything we need to predict any future values of the motion. Looking back we asked "How far does it go?" which is another way of saying "where does it land?"

We ponder this and eventually realize that "Where does it land?" is asking for the final position. When it lands it is back on the ground at $y = 0$. Therefore we should set $y_f = 0$. However, we don't know x_f. Can we make progress? Let's update our equations and see.

When we plug in $y_f = 0$ then we have

$$y_f = 15\sqrt{2}t - 5t^2 = t(15\sqrt{2} - 5t) = 0$$

We can solve this for t and then we will know how much time has elapsed since we kicked the ball until it hits the ground.

$$t(15\sqrt{2} - 5t) = 0$$

$t = 0$ is a solution: well that makes sense because at the start the ball was at $y = 0$

the second solution is given by

$$15\sqrt{2} - 5t = 0 \Rightarrow t = \frac{15\sqrt{2}}{5} = 3\sqrt{2}$$

This says the ball traveled for $3\sqrt{2}$ seconds before landing on the ground ($y_f = 0$). We ponder a bit and realize that the equation for x_f is:

$$x_f = 15\sqrt{2}t$$

If we plug in $t = 3\sqrt{2}$ then this should give us the distance the ball has traveled: plugging in we find that the ball lands at

$$x_f = 90$$

in meters.

Notice that we solved the equation in y in order to get something about x. In math this is called a "system of equations" and they are EVERYWHERE! In particular these are called "parametric equations" since both depend on the variable t.

Now a tricky demonstration: I am going to solve y_f as a function of x: $y(x)$. This will give us the PATH the ball takes in SPACE - what we see with our eyes. This will tell us nothing about time per se but it is satisfying to prove that the path MUST be parabolic in space.

To begin we consider only constant acceleration in the y-direction due to gravity.

$$a_x = 0 \qquad a_y = g = -10$$

We will replace a_y with g as is tradition.

Now,

$$x_f = x_i + v_{ix}t$$
$$y_f = y_i + v_{iy}t + \frac{1}{2}gt^2$$

I shall solve (1) for t and then plug that t into the y equation thereby eliminating t and yielding $y(x)$

$$t = \frac{x_f - x_i}{v_{ix}} = \frac{\Delta x}{v_{ix}}$$

now in the y equation everywhere I see t I plug in $\frac{\Delta x}{v_{ix}}$

$$y_f = y_i + \frac{v_{iy}}{v_{ix}}(x_f - x_i) + \frac{g}{2}\left[\frac{x_f - x_i}{v_{ix}}\right]^2$$

Goodness, what a mess! Let us think carefully.

Now, y_i, x_i, v_{ix}, v_{iy} are all just numbers. They are initial values and we consider them knowns. If they are known numbers then $\frac{v_{iy}}{v_{ix}}$ is some number as well. So we have something like

$$y = ax^2 + bx + c \quad \text{ with } a < 0$$

for $g = -10$ we have

$$y_f = -\frac{5}{v_{ix}}(x_f - x_i)^2 + \frac{v_{iy}}{v_{ix}}(x_f - x_i) + y_i$$

In this form $(x_f - x_i)$ can be considered a variable: y depends upon it. We can plug in values from our previous example and we get

$$y_f = x_f - \frac{x_f^2}{90}$$

The general equation $y(x)$ is therefore the form of a parabola and so projectiles in two dimensions should follow a parabolic path in space. Go kick a ball, you'll see what I mean!

Now, in our ball example, let's ask "How high does it go?

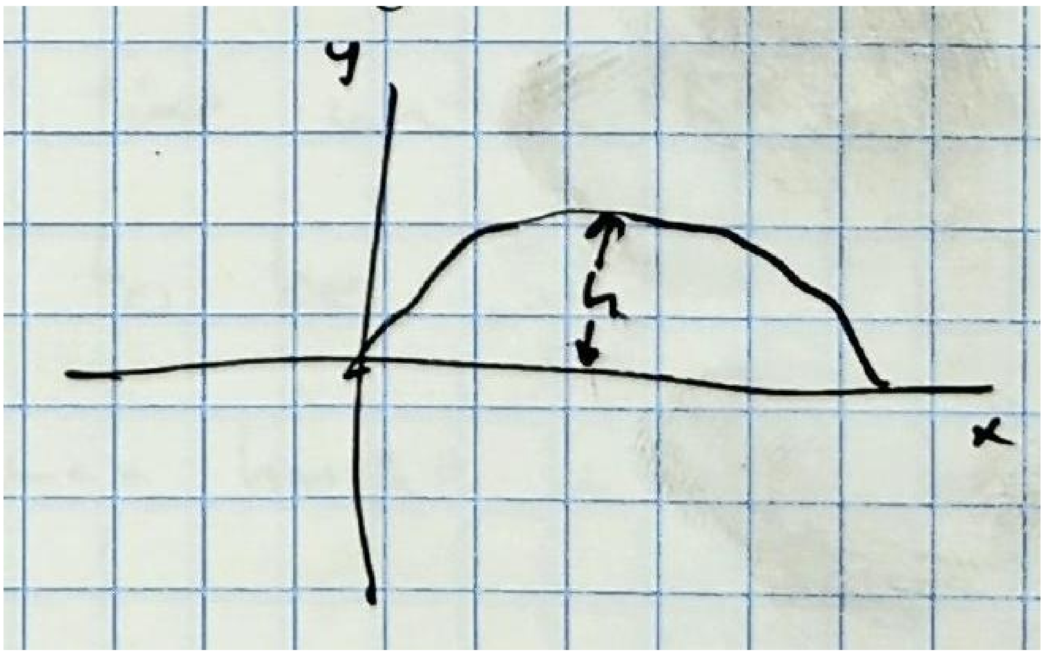
This is quite tricky but recall what we found in 1D. In one dimension the max height was where $v = 0$

Does this trick work in two dimensions? Yes and No.

If we think long enough we come to realize that the motion in y direction is all that matters for this question.

The maximum height is where the motion stops going up and starts going down! There may still be a velocity along x but at the max height the velocity along y is assuredly zero. Let's call $y_{max} = h$ and go back to our four main equations:

$$x_f = 15\sqrt{2}t$$
$$y_f = 15\sqrt{2}t - 5t^2$$
$$v_{fx} = 15\sqrt{2}$$
$$v_{fy} = 15\sqrt{2} - 10t$$



There are a few routes to solution
(1) plug in $y_f = h \quad v_{fy} = 0$

$$y_f = h = 15\sqrt{2}t - 5t^2$$
$$v_{fy} = 0 = 15\sqrt{2} - 10t$$

The simplest route in this method is to solve for t from $v_{fy} = 0$ and then plug into y_f

Well,

$$0 = 15\sqrt{2} - 10t \Rightarrow t = \frac{3}{2}\sqrt{2}$$

then

$$y_f = h = 15\sqrt{2}\left(\frac{3}{2}\sqrt{2}\right) - 5\left(\frac{3}{2}\sqrt{2}\right)^2$$

$$= 45 - 5\left(\frac{9}{4}\cdot 2\right) = 45 - \frac{45}{2} = 22.5 \text{ m}$$

This is the maximum height!

(2) the more beautiful route is to say:

1. I discovered the path in space is a downward parabola and these are symmetric about the vertex/maximum.
2. I previously found the time until the ball hits the ground to be $3\sqrt{2}$ s
3. therefore the time until the max height is $\frac{3\sqrt{2}}{2}$ seconds!
4. plug this into $y_f = h$ and solve.

We might also ask "above what point on x-axis is the max height?

We again leverage symmetry: If the max distance along x is 90 m then the max height occurs at $x = 45$ m

You may have noticed that we never used the 1D eg

$$v_f^2 = v_i^2 + 2a\left(x_f - x_i\right)$$

but we can!

$$v_{fx}^2 = v_{ix}^2 + 2a_x\left(x_f - x_i\right)$$
$$v_{fy}^2 = v_{iy}^2 + 2a_y\left(y_f - y_i\right)$$

These can simplify computations greatly! For example, the max height?

$$v_{fy}^2 = v_{iy}^2 - 20(h-0) = (15\sqrt{2})^2 - 20h$$

but at $y_f = h$ we have $v_{fy} = 0$

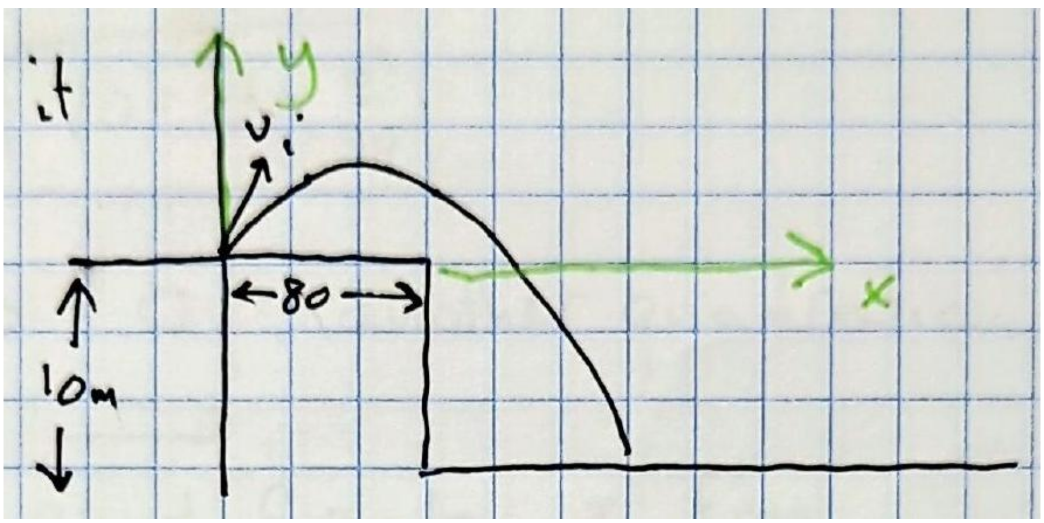
$$0 = 450 - 20h \Rightarrow h = 22.5$$

Notice for projectiles that if $a_x = 0 \Rightarrow v_{fx}^2 = v_{ix}^2 - 0 \Rightarrow v_{fx} = v_{ix}$ the velocity along x never changes!

In general $y_{\max} = h = \frac{v_{iy}^2}{2g}$

Q4: What if the edge of a cliff 10 meters tall and is 80 meters from where you kick?

Step 1: Draw it



Be advised: folks often don't formulate the questions they ask in a non-ambiguous way. Often the best we can do is "interpret" what the asking party is trying to convey linguistically. Solving physics problems in class is part mind-reading! Solving problems out in the real world you're free to interpret you own questions however you may feel you should.

Okay, so we know the ball will go off the cliff because we previously found the max distance (under this initial kick) to be 90 meters. This time we want the max x-position and therefore we follow the same logic as previously. Notice however, that because the y-position is changing the ball travels for a longer amount of time. During its fall below the cliff it is still traveling with non-zero v_x so we can expect it will cover more horizontal distance in this situation.

The final y-position is at the bottom of the gulch at $y_f = -10$ meters according to our chosen coordinate system. It is negative because we are initially at $y_i = x_i = 0$.

We have many ways to solve this question. Step 2 is choose coordinates which we did. Step 3 is write out what we know; which we did. Step 4 is phone a friend, Galileo, and step 5 is plug it all into Galileo's equations.

Writing out the x, y positions serperately

$$\begin{aligned} & x_f = x_i + v_i t = 0 + 15\sqrt{2}t \\ & y_f = -10 = y_i + v_i t - 5t^2 = 15\sqrt{2}t - 5t^2 \\ \therefore \quad & -5t^2 + 15\sqrt{2}t + 10 = 0 \end{aligned}$$

This is a quadratic equation and therefore we can solve it for t yielding the time it takes to arrive at $y_f = -10$. Then we could plug this time into our equation for x_f and bing, bang, boom, we find the final position of our ball.

But we have a second way. We previously fond y_f as a function of x_f

$$y_f\left(x_f\right) = x_f - \frac{x_f^2}{90} \quad \text{but } y_f = -10 \text{ m}$$

Call the final position $x_f = d$

Then

$$y_f(d) = -10 = d - \frac{d^2}{90}$$

Or

$$-\frac{d^2}{90} + d + 10 = 0$$

Yet another quadratic!

Using the quadratic formula we find

$$d = -9.09 \text{ m or } 100 \text{ m}$$

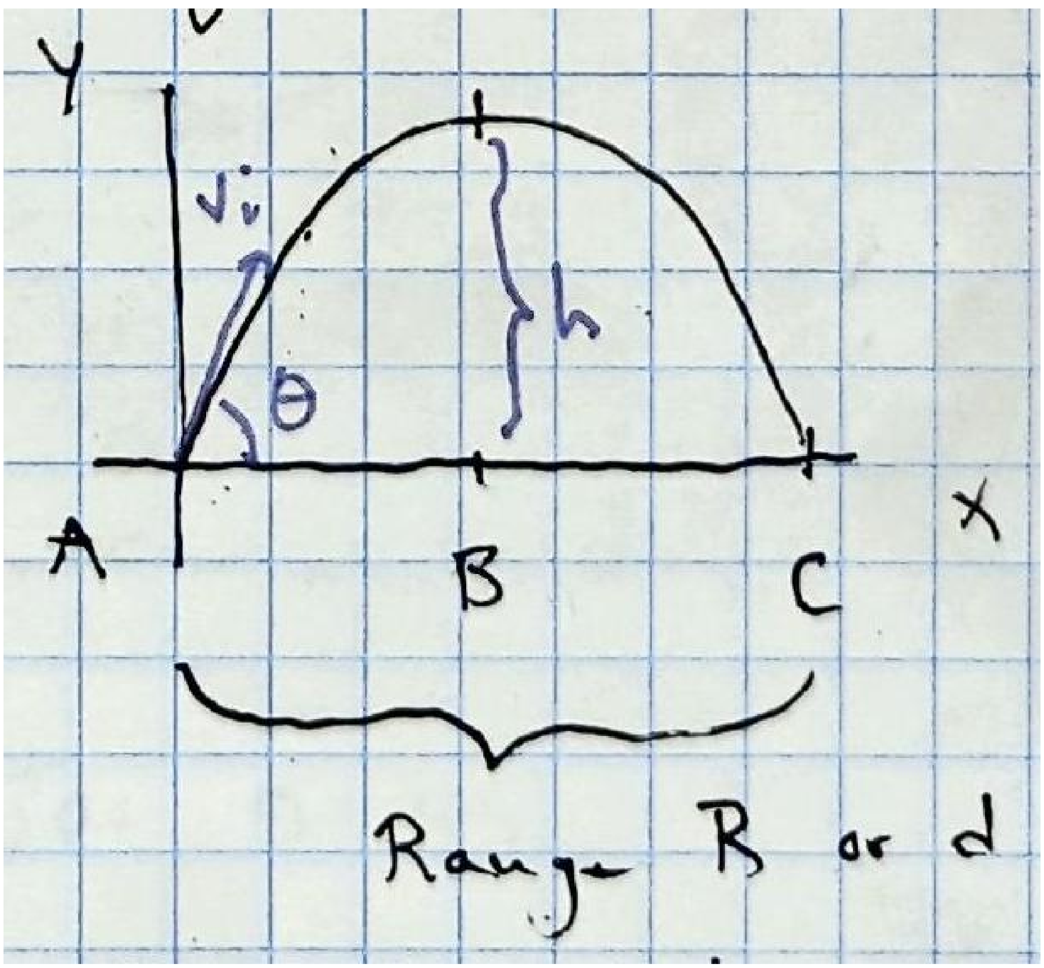
Since we kick the ball in the positive x direction we had better hope the range isn't negative (i.e. it flies behind us)!

The only physically reasonable solution therefore is

$$d = 100 \text{ m}$$

Either route yields $d = 100$. One either solves a quadratic for t or for d. All roads lead to Rome. The more ways you know how to solve a problem the more skilled you are at problem solving.

Now, it turns out there is a general formula for range that depends on the initial "speed" (length of velocity vector!) and direction such that you can mindlessly plug-in your initial data. Choose coordinates to have the ball at $(0, 0)$. The solution, turns out to be completely independent of the coordinates you choose.



Recall how we found the time of flight t_{max}. We solve

$$y_f = 0 = y_i + v_{iy}t - \frac{1}{2}gt^2$$

Plugging in our known quantities $g = -10$ and the initial positions we have

$$y_f = v_{iy}t - 5t^2$$

Generally, t_{max} occurs at $y_f = 0$ so that we may solve for t

$$y_f = 0 = v_{iy}t_{max} - 5t_{max}^2$$

then

$$t_{max} = \frac{2v_{iy}}{a_y}$$
$$t_{max} = \frac{2v_{iy}}{10} = \frac{v_{iy}}{5}$$

Well this is the time of flight. We can now find the range $x_f = R$ by plugging in what we found for t_{max}.

$$\begin{aligned} x_f &= x_i + v_{ix}t + \frac{1}{2}a_xt^2 \quad a_x = 0 \\ &= v_{ix}t \\ \Rightarrow R &= \frac{2v_{ix} \cdot v_{iy}}{10} \end{aligned}$$

But

$$v_{ix} = |v_i| \cos\theta$$

$$v_{iy} = |v_i| \sin\theta$$

Then,

$$d = \frac{2v_i^2 \cos\theta \sin\theta}{g}$$

However, using a trig identity we can also write this as:

$$d = R = \frac{v_i^2}{g}\sin(2\theta)$$

This is the form one usually sees the range of a projectile written in.

In our case $g = 10, v_i = 30, \theta = 45$ so that, plugging in,

$$R = \frac{900}{10}\sin 90 = 90 \text{ m}$$

Notice the max value of $\sin\theta$ is at $90°$ so choosing $\theta = 45°$ gives the maximum possible range!

Dear reader, do not attempt to use this equation for a projectile off of a cliff. The formula ONLY works when $y_i = y_f$, i.e. on flat ground.

If we are on or below a cliff then a more complicated expression applies; I encourage you to try your hand at deriving it.

$$R = \frac{v_i}{g}\cos\theta\left[v_i\sin\theta + \sqrt{v_i^2\sin^2\theta + 2gh}\right]$$

Notice that This reduces to the form we found for $y_i = y_f$ above.

Q5: How long does the ball travel for?

Well we found the range $x_f = R$. Then following Galileo we have

$$x_f = R = 100 = 15\sqrt{2}t$$
$$\therefore t = \frac{100}{15}\sqrt{100} = 4.75$$

Q6: What is the maximum height of the ball relative to the bottom of the cliff?

Well, we previously found the max height of level ground so the max height relative to the bottom of the cliff must be $h = 22.5 + 10 = 32.5$m. If you always have a sketch handy a lot of these questions work themselves out!

Often you will be ask questions like "What angle must you jump to clear a hurdle?"

In these cases you have the max height (the hurdle height) and then you'll need to solve for velocities and/or angles.

The maximum height, in the case of $y_i = y_f$ is given by

$$h = \frac{v_i^2 \sin^2 \theta}{2g}$$

If you know the initial speed and the hurdle height then you can solve for θ using the standard algebraic methods. That's it! Now go forth and practice - you will NOT master ths topic unless you work through many complicated problems!

Forces and Newton

You have now covered most of what was known, generally, in Physics up until roughly 1650. The main tools had been given by Euclid, Pythagoras, Al Jabr, Aristotle, and Galileo. In parallel to all this a new tool was slowly being developed that would revolutionize the world and cement confusion abound for centuries regarding infinity. The tool was calculus.

Isaac Newton played a MAJOR role and justifiably is considered one of the greatest minds to have existed (that happened to be wealthy enough to spend time thinking! There has surely been equal or even greater minds to have existed that were forever lost in the dustbin of poverty).

Newton set several new paths of thought but we will mainly concern ourselves with just three.

1. He independently co-discovered the calculus
2. He formulated 3 general laws of motion
3. He developed a theory of Gravity

We will only mention calculus in passing. It's my humble opinion that everyone should learn it. Modern and Classical physics would be (almost) impossible without it! For now, however, we will discuss number 2) Newton's Laws.

You may have found it curious that we could describe 1D and 2D kinematics/motion without mentioning the concept of mass whatsoever!

As far as we know ALL matter (anything that taker up space and time) comes with a little label we call mass. We all have an intuition of what mass is yet it is a surprisingly complex thing and, as it happens, the only reason you have this intuition is because you grew up in a time and place where humanity has discussed mass for centuries. But there was a time when no-one had any intuition about mass whatsoever.

For our purpose we will define it as "a quantity of master" and we will call it "m" (not to be confused with meters)!

$$m \equiv \text{ a quantity of matter}$$

This is a frustratingly imprecise definition. In fact people still spend their entire lives discussing what the heck mass even is or isn't! My preference is to imagine a name tag or label everything carries around the universe.

The units of mass we will use are the kilogram (kg).

Be WARNED people who have NOT taken physics will use the words "mass" and "weight" interchangeably. E.g: "I weigh 75kg". In physics this is WRONG WRONG WRONG.

We shall be MUCH more careful than that!

Let un define what we mean by "weight".

Definition

$$\vec{w} = m\vec{g}$$

where $|\vec{g}| = 9.8\ m/s^2$

This is actually what we measure on our bathroom scale: not the mass!

However, the scale divides out g automatically for us such that $m = w/g$. Because of this if you took your scale to the moon it would be completely wrong because $g_{moon} < g_{earth}$

I have a mas of 75 ± 3 kg most days and this is ALWAYS true whether I'm on earth, the moon, or near a black hole: It is MY mass, MY label. If I take my scale to the moon it will claim my mass is 50 kg! But my label is NOT 50kg, it is 75! The issue is the scale's reading is calibrated to the earth. This is a fantastically important point to appreciate in physics: your mass doesn't change unless you eat or excrete stuff! And it certainly doesn't change (appreciably) if you're slowly moving from one place to another. (in order to be correct I use the word "appreciably". There is an effect due to Einstein that find mass can vary as one moves ever closer to the speed of light. We shall not consider such beautiful complexities here.)

Now, we may have a formula for weight but we haven't said what it is. It has really weird units of

$$[w] = \frac{\text{mass meters}}{\text{second }^2} = \frac{kg \cdot m}{s^2}$$

A quantity with this specific combination of units shall be called a "Force" and rather than carry all these kilograms, meters, and seconds around we give a new name to this unit called a "Newton".

Okay, so we call weight a type of "force". But what the heck is a "Force"? We shall think about "force" in two main ways.

1. a push or a pull on an object

2. anything that changes an objects velocity

The 2^{nd} definition is preferable by far but the 1^{st} is more intuitive. Now, a Force is a vector quantity; it has direction and magnitude/length.

Eg." I push the wall to the left with all my strength" is MUCH different than pushing "up".

We will work with several different Forces and shall write it as $\vec{F}$.

We have that weight is a force which is a vector, which has units of Newton (N), which is the same as the units of mass times acceleration. Or, $w = mg$. Perhaps all forces are related to mass time acceleration?

Before we answer this question I want to define one more thing to set us up. Galileo gave us the concept of "Inertia" but it had previously been defined by the philosopher Mozi in China sometime around 475-221 BC.

Inertia is a resistance to a change in velocity.

Galileo postulated that all matter has inertia and something would stay moving in a straight line forever unless "Something" changes its velocity.

This is an ABSOLUTELY BREATHTAKING and BOLD abstraction, Think about it:

Nothing on earth moves at a constant velocity forever. Nothing! Gravity (which we haven't covered) constantly increases an objects velocity until it is stopped by the ground. But Galileo is arguing that if there were no gravity or air or what-have-you then a mass would just keep on going at a constant velocity. How bizarre! Again, you dear reader, grew up in the space zeitgeist and objects traveling in straight lines forever is probably intuitive to you.

Anyways, this was a postulate: there was no evidence for it whatsoever but he reasoned that it must be true. Brilliant. That's why he earns the big bucks.

So, something with more inertia is harder to move than something with less. what we've been calling "mass" should properly be called "inertial mass". It's a picky but important distinction to make as we'll see later but realistically we assume that ALL mass is inertial mass.

I will define one last new concept but we wont deal with it until much later.

Momentum:

$$\vec{p} = m\vec{v}$$

mass times velocity, it has no specially named units, just

$$[p] = \text{kg} \cdot \frac{m}{s} = N \cdot s$$

You are now prepared for a Kuhnian paradigm shift in our description of the nature of reality. The notions seem simple enough but try to put yourself in Europe in the 1600s. The enlightenment was about 100 years old and had slowly spread from Italy to England. The Church (Rome) was arguably the most powerful institution in the world and the church of England was only $50-75$ years old. Galileo was almost executed for professing what he knew to be true ("Yet it moves") and a new science was questioning old knowledge of the Greeks. In this context Newton wrote his three Laws of Nature:

1. an object at some velocity stays at that same velocity unless a force acts on it
2. $\vec{F} = m\vec{a}$: A change in velocity (acceleration) is caused by a Force
3. If an object exerts a Force on another object then the 1^{st} object experiences an equal and opposite force, viz, $\vec{F}_{12} = -\vec{F}_{21}$

We will use these three laws to solve a variety of problems. We must be a bit careful with $\vec{F} = m\vec{a}$

This is both an idea and a mathematical formula. By "idea" I mean that the two sides of the equals sign mean completely different things that happen to be equal

A force is conceptually different than a mass and acceleration
(2) says that If a force acts on m then it will cause an acceleration of $\vec{a} = \vec{F}/m$. If an object has an acceleration then there must be a Force $\vec{F} = m\vec{a}$ involved

(2) is a formula for how to find $\vec{F}$ or $\vec{a}$ as vectors. If there are no forces then $\vec{a} = 0 = \frac{\Delta\vec{v}}{\Delta t} \Rightarrow \vec{v}$ constant. So, we have all three new ideas and concepts swirling about and the question remains "How can we leverage these?"

This leads us to a tool we will be using constantly: the Free Body Diagram (FBD).

You have ample practice with vectors and scalars. Forces are vectors. So suppose you have an object of mass m near the earth on the ground. Then, since it's near the Earth it has a force called the weight

$$\vec{w} = m\vec{g}$$

This vector points down towards the center of the Earth. We usually define it in a Cartesian coordinate system with y being the vertical axis. In this case the weight vector would be written as

$$\vec{w} = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

There are two objects here: the ball and the earth.

3. says that the farce of the ball on the earth equals the opposite of the force of the Earth on the ball

$$\vec{F}_{b\cdot E} = -\vec{F}_{E\cdot b}$$

The force of the ball on the earth is the weight so the equal and opposite $\vec{F}_{E\cdot b} = -\vec{w}$

We call this "reaction" force the "Normal" Force. $\vec{N} = -\vec{w}$

IMPORTANT: THE NORMAL FORCE IS ALWAYS PERPENDICULAR TO THE SURFACE.

Let's think about what Newton tells us and what the ball is doing sitting there lazily on the ground

Well, if the ball is resting on the earth the it is assuredly not moving, which means $\vec{v} = 0$.

Between now ($t = t_i$) and later ($t = t_f$) the velocity vector is still zero. So certainly we have $\Delta\vec{v} = 0$

But 2) says that if $\Delta\vec{v} = 0$ then the total force $\vec{F}_{total} = 0$. So we ask "what are all the forces acting on our ball?"

We have just two forces $\vec{w}$ and $\vec{N} = -\vec{w}$

Since Force is a vector we can add up Forces to find the resultant $\vec{F}_{total} = \vec{F}_1 + \vec{F}_2$ etc.

Here we have $\vec{w}$ and $\vec{N}$

So $\vec{F}_{total} = \vec{w} + \vec{N} = \vec{w} - \vec{w} = 0$

The total Force is zero and therefore the object is not accelerating: $\vec{N}$ perfectly balances $\vec{w}$.

But if we suddenly opened a trap door at $t = 5$ seconds. Then suddenly there is no more $\vec{N}$ and the will certainly start falling, i.e. accelerating down the newly available hole: there is no more Normal force to keep the ball still. Then we have,

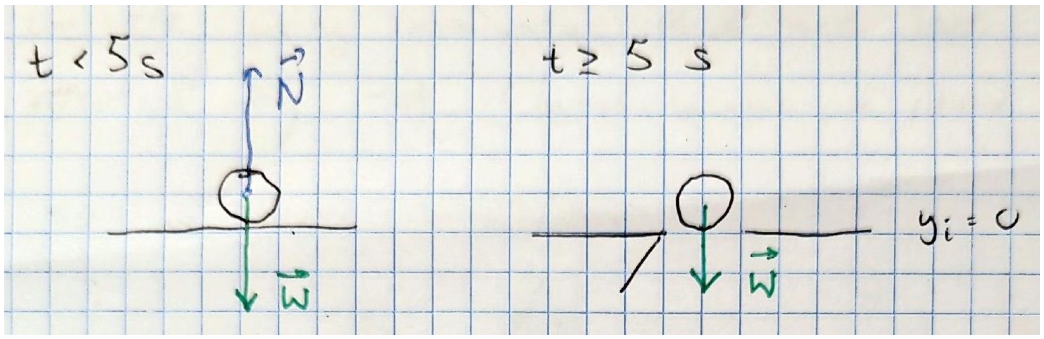
$$F_{total\cdot y} = w_y = -mg$$

Nothing is happening in the x direction so we study only the y-components. This equation says the ball accelerates with $a = g = -10m/s^2$

But from 1D kinematics we know

$$y_f = y_i + v_i t + \frac{1}{2}at^2$$

so that $y_f = -5t^2$. Our FBD is sketched below for before and after the trapdoor opens.



FBDs help us to set up the equation $\vec{F}_{\text{tot}} = m\vec{a}$. We simply try to imagine all of the forces $\vec{F}_i$ and which way they point and then sketch them and finally decompose them into $x - y$ coordinates.

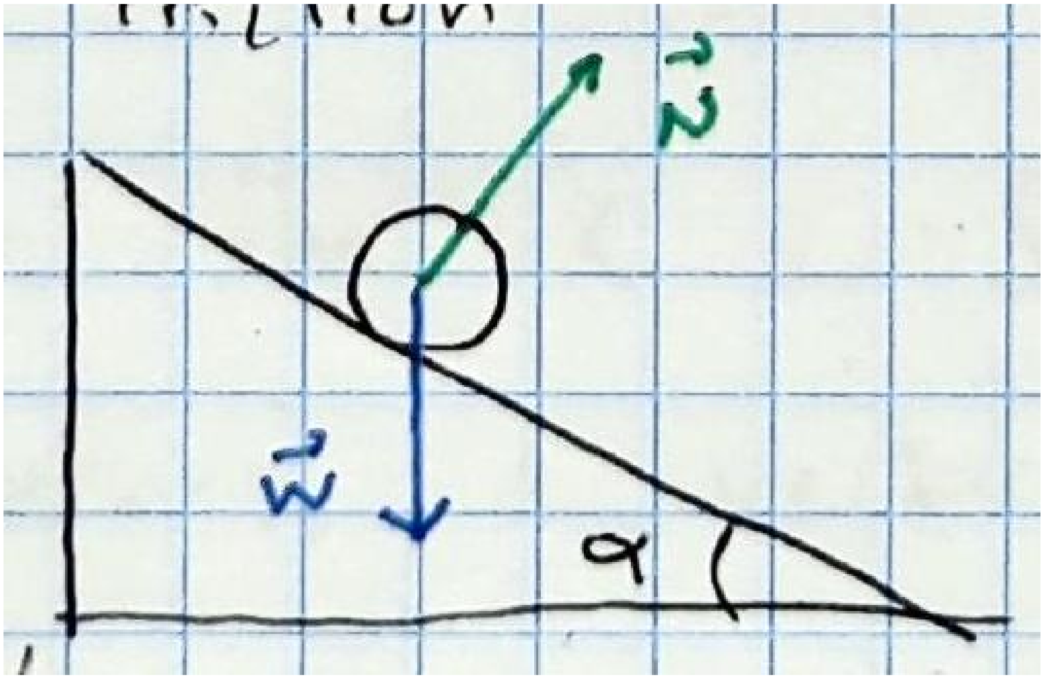
There may be any number of complex forces involved in a system but our strategy is always the same: draw the FBD, choose coordinates, decompose forces into cartesian components and then add up all the y-components, add up all the x-components and set these equal to ma_x and ma_y.

Example: A block sliding down an incline with no friction

Our goal is to ask what Newton has to say. Specifically, we will solve for the acceleration of the block and maybe even solve for the normal force.

Again, our strategy is the same except now we have an added step of the free body diagram.

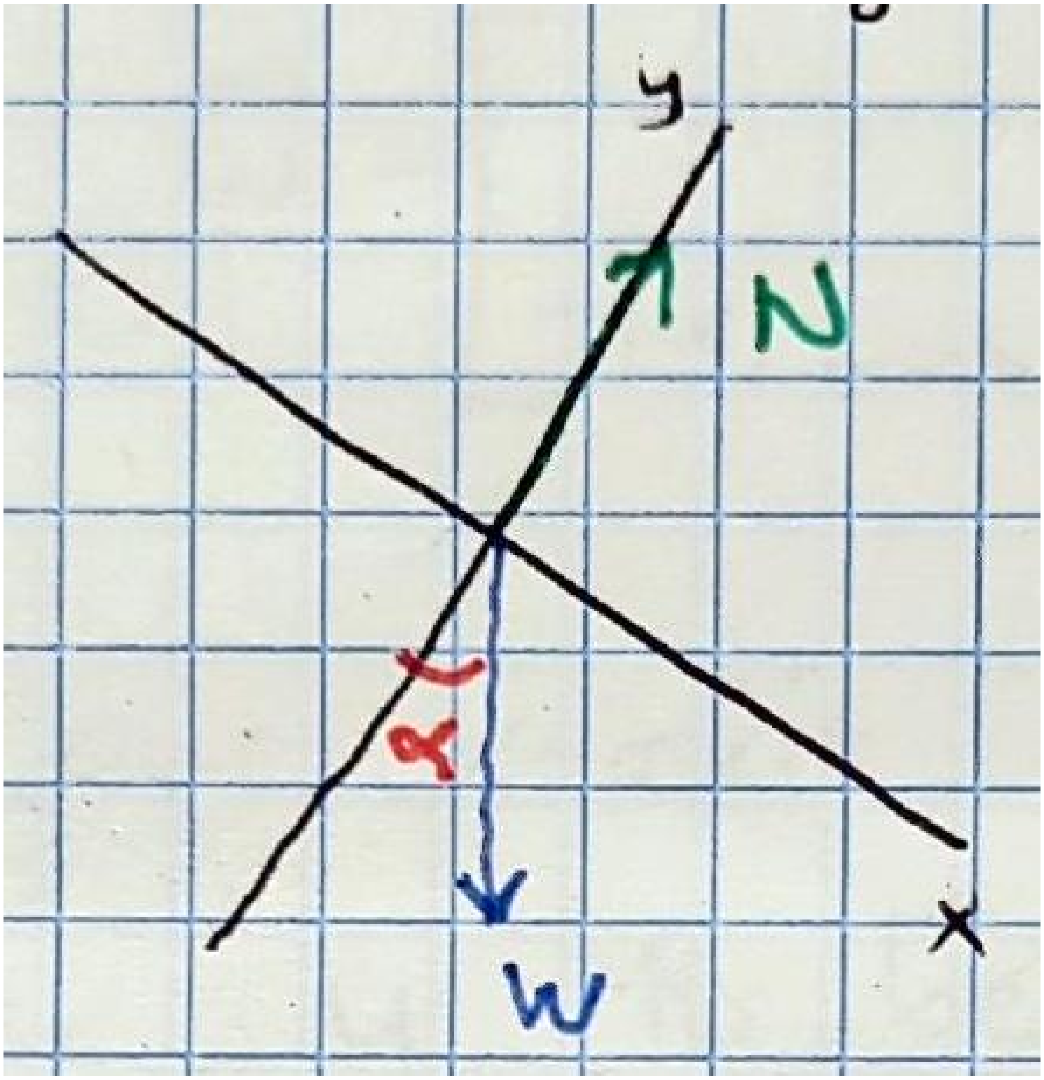
Step 1: Draw it
Step 2: Free body diagram (draw all forces and their directions). See below.



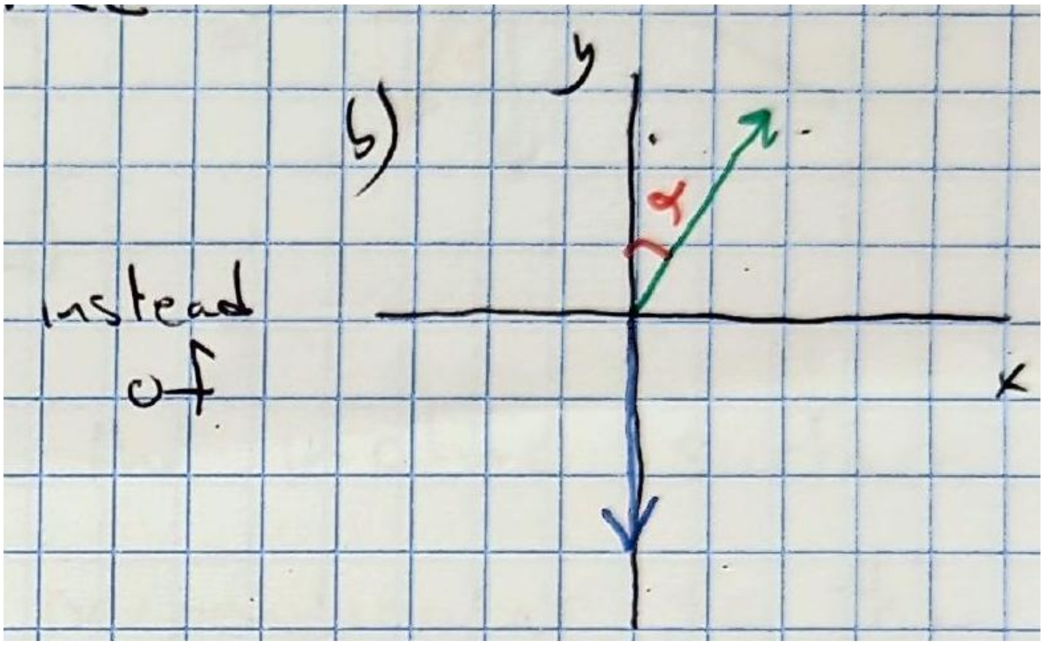
Step 3: This is a 2D problem so we need to use x, y coordinates.

This is the hardest part of solving these types of problems. What coordinates shall we use? Our general rule will be to use coordinates such that the x, y axes are lined up with as many of the forces as possible. Here we have two choices: align y with the Normal force or align y with the weight vector?

We shall choose to line up with the Normal Force in order to demonstrate a new way of thinking about coordinates.



instead of



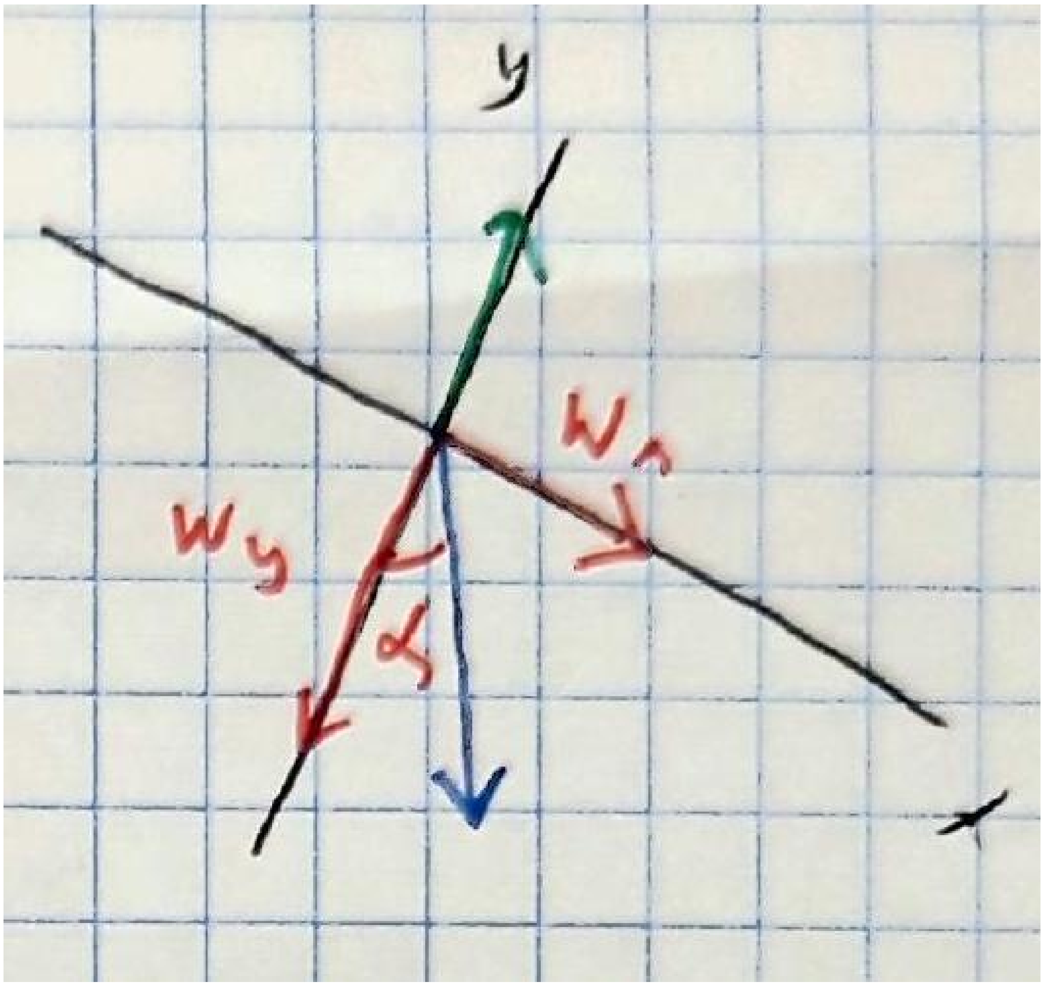
This is admittedly kind of weird. It is as though we are using coordinates that have been rotated by $-\alpha$ (minus because we rotated that way.

Okay, we have vectors on a coordinate system so our next step is to decompose all the vectors in this chosen coordinate system.

$\vec{N}$ is aligned with the y-axis and has O component in our x direction, and all of its length in the $+y$ direction.

$$\vec{N} = \begin{bmatrix} 0 \\ N \end{bmatrix}$$

Now, what are the x, y components of $\vec{w}$? we are decomposing a vector that looks like below



so then we have

$$\vec{w} = \begin{bmatrix} |w|\sin\alpha \\ -|w|\cos\alpha \end{bmatrix}$$

w_y is negative because in our chosen cords it points down; w_x is positive and points to our right. Furthermore, we know that $|w| = mg$ ($g = 10$ no minus sign!!). Therefore we have

$$\vec{w} = \begin{bmatrix} mg\sin\alpha \\ -mg\cos\alpha \end{bmatrix}$$

$$\vec{N} = \begin{bmatrix} 0 \\ N \end{bmatrix}$$

Next step: we call up Newton and he says "add up all the forces and set them equal to mass times acceleration"

$$\sum \vec{F}_i = \vec{F}_1 + \vec{F}_2 = \vec{w} + \vec{N} = m\vec{a}$$

here i stands for force 1, force 2, force 3, etc

Well, let's fill things in. we have

$$\begin{bmatrix} mg\sin\alpha \\ -mg\cos\alpha \end{bmatrix} + \begin{bmatrix} 0 \\ N \end{bmatrix} = m \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

Notice that our x-axis is the incline surface itself and we chose y to be perpendicular to the incline.

We ask, "is there motion along x in this scenario?" Yes, the block should be sliding down.

"Is there motion to be expected along y?" No! If there were then that would mean the block is either floating away off the incline or it would be falling through the incline. That would be weird indeed. We could study such things, however we will assume it only slides downward along x and not at all along our chosen y direction.

Therefore, we demand that $a_y = 0$. We may now fill this in

$$\begin{bmatrix} mg\sin\alpha \\ -mg\cos\alpha + N \end{bmatrix} = m \begin{bmatrix} a_x \\ 0 \end{bmatrix}$$

Let's take stock: We have two equations (top slot and bottom slot) and several unknowns. Suppose we knew the mass and α, the angle of the incline. Then we would have just two unknown quantities: a_x and N. Two equations, two unknowns: we can solve this! Writing these out we have

in the x direction

$$mg\sin\alpha = ma_x \Rightarrow a_x = g\sin\alpha$$

Next, in the y direction

$$N - mg\cos\alpha = 0 \Rightarrow N = mg\cos\alpha$$

Problem solved! we found both the acceleration AND the normal force. Given this a_x would could ask Galileo how the block slides down and in his two equations we could plug in this value of acceleration to determine the positions and velocities as a function of time.

Shockingly the mass cancels (this is subtle!) and the block's acceleration is not g but rather is reduced by $\sin\alpha$.

Does this make sense? Well, if $\alpha = 90°$ then the ball is just falling at g and if $\alpha = 0$ then $\sin 0 = 0$ and $a = 0$ because there is no more force in the x-direction. The theory works!

Solving $F = ma$ problems

Let's proceed by example.

Q1: Suppose I push a box with a force of $8N$. If the mass of the box is 1 kg then what will be the box's acceleration?

$$\vec{F} = m\vec{a}$$

We simply plug in our data and solve

$$a = 8m/s^2$$

Q2: Now suppose the box is $10,000$ kg !

then

$$8 = 10,000\vec{a} \Rightarrow a = 8 \times 10^{-4} m/s^2$$

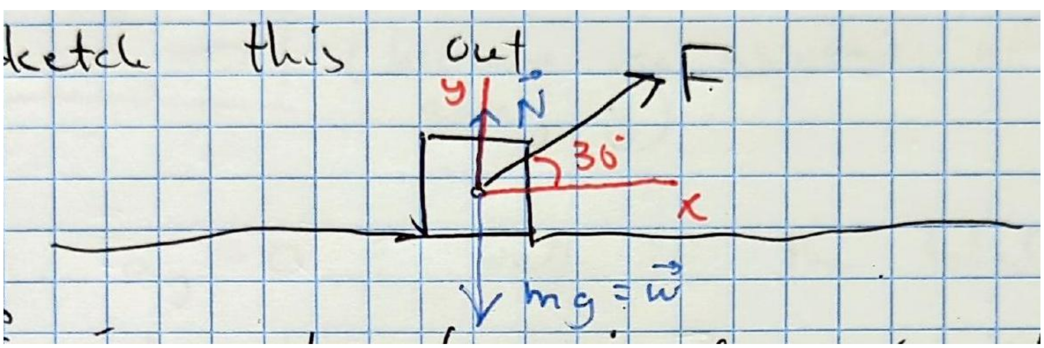
We find that a heavier box accelerates less. Duh!

Q3) Suppose I pull a 2kg object with a force of 3 N at $30°$ with respect to the floor such that it doesnt lift off the ground. Find the acceleration

Here the force is in 2 dimensions and we are given its length and direction (3 and $30°$ respectively). Then

$$\vec{F} = 3N \text{ at } 30°$$

Step 1, 2, and 3: Let's sketch this out with the forces and chosen coordinates labeled.



Step 4: Since $\vec{F}$ is a two dimensional vector let us find its x y components in the coordinates we have chosen.

Recall that we have previously found that for an arbitrary vector $\vec{a}$ that

$$a_x = |a| \cos\theta$$

$$a_y = |a| \sin\theta$$

Plugging in $|F|$ and $\theta = 30°$ we have

$$F_x = 3\cos 30 = \frac{3\sqrt{3}}{2}$$
$$F_y = 3\sin 30 = \frac{3}{2}$$

we also have a weight vector $\vec{w}$ along the y direction.

If the object stays on the ground then this means that $N \neq 0$. N would be zero if the box no longer made contact with the ground. If i pull hard enough, even at 30 degrees I could potentially lift the box up along y.

Let us demand that $a_y = 0$ - that is to say we demand that the box doesn't lift up. If we do this then we might be able to find a Normal force vector and if our pulling force in the y direction is not greater than this condition then the box will remain on the ground.

Therefore, we have, according to Newton

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} + \begin{bmatrix} 0 \\ N \end{bmatrix} + \begin{bmatrix} 0 \\ -mg \end{bmatrix} = m \begin{bmatrix} a_x \\ 0 \end{bmatrix}$$

plugging in we have,

$$\begin{bmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ N \end{bmatrix} + \begin{bmatrix} 0 \\ -20 \end{bmatrix} = 2 \begin{bmatrix} a_x \\ 0 \end{bmatrix}$$

So that we may solve two equations

$$\begin{bmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} + N - 20 \end{bmatrix} = 2 \begin{bmatrix} a_x \\ 0 \end{bmatrix}$$

Then, we find, since $\frac{3}{2} < 20$ then

$$N = \frac{37}{2}$$

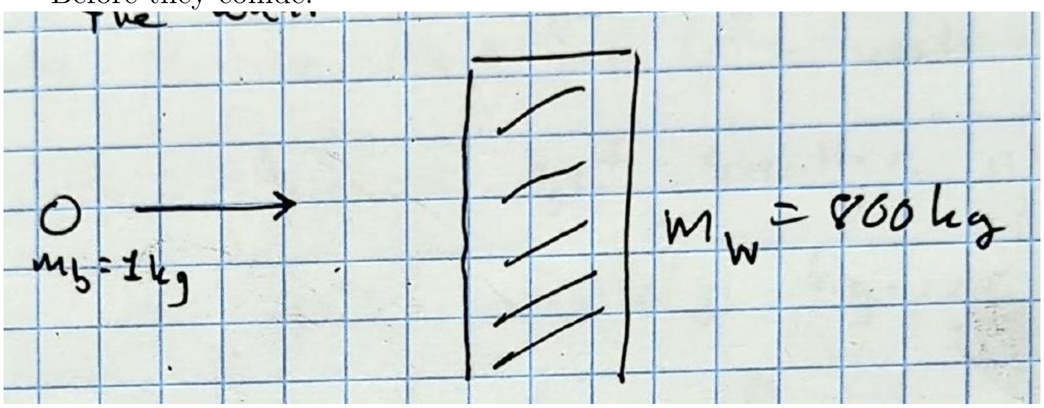
If $F_x > 20$ then our box would accelerate upwards.
We also find

$$a_x = \frac{3\sqrt{3}}{4}$$

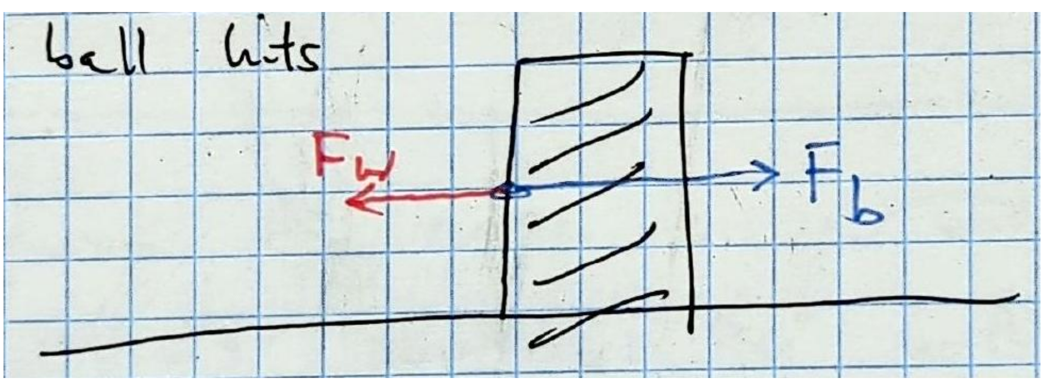
Q4: Suppose I throw a 1 kg ball against a VERY massive wall. Find the acceleration of the ball and the wall if $m_{wall} = 800$ kg given that the ball exerts 3 Newtons on the wall.

1. sketch!

Before they collide:



During the collision:



Here the key is both the second and third laws of Newton. Let us write F_bW for the force the ball exerts on the wall and F_Wb for the force the wall exerts on the ball.

When the ball applies its force of $3N$ on the wall, the wall in response applies an equal is opposite force on the ball. But since we know the magnitude of the force then we can plug into Newton's second law as

$$F_{Wb} = -m_b a_b$$

$$= -F_{bW} = m_w a_w$$

Plug in values and solve for a_b and a_W to find

$$a_W = +\frac{3 \text{ N}}{800 \text{ kg}} = 0.004 \text{ m/s}^2$$
$$a_b = -\frac{3 \text{ N}}{1 \text{ kg}} = -3 \text{ m/s}^2$$

Notice the directions are crucial here! This result makes sense. The wall ever so slightly accelerates in the $+x$ direction and the ball bounces back in the $-x$ direction with greater acceleration than the wall! We notice that if the wall were heavier then a_W would get even smaller. As the mass of the wall gets infinitely bigger relative to the ball the acceleration of the wall goes to zero.

This is precisely what we observe in everyday life: balls bounce off walls. Interestingly, if you jump and land on the Earth, this results says that the Earth accelerates backwards by an ever so tiny amount! How many people must all land on the earth in the same place in order to impart the acceleration we obtained for the wall (take their force to be their collective weights)?

Q5: Suppose a horse is in space with two rockets pointing $90°$ with respect to each other. One applies a force of 100 N up the other applies 50N to the right of the horse. If the mass of the horse is 1000 kg find the acceleration.

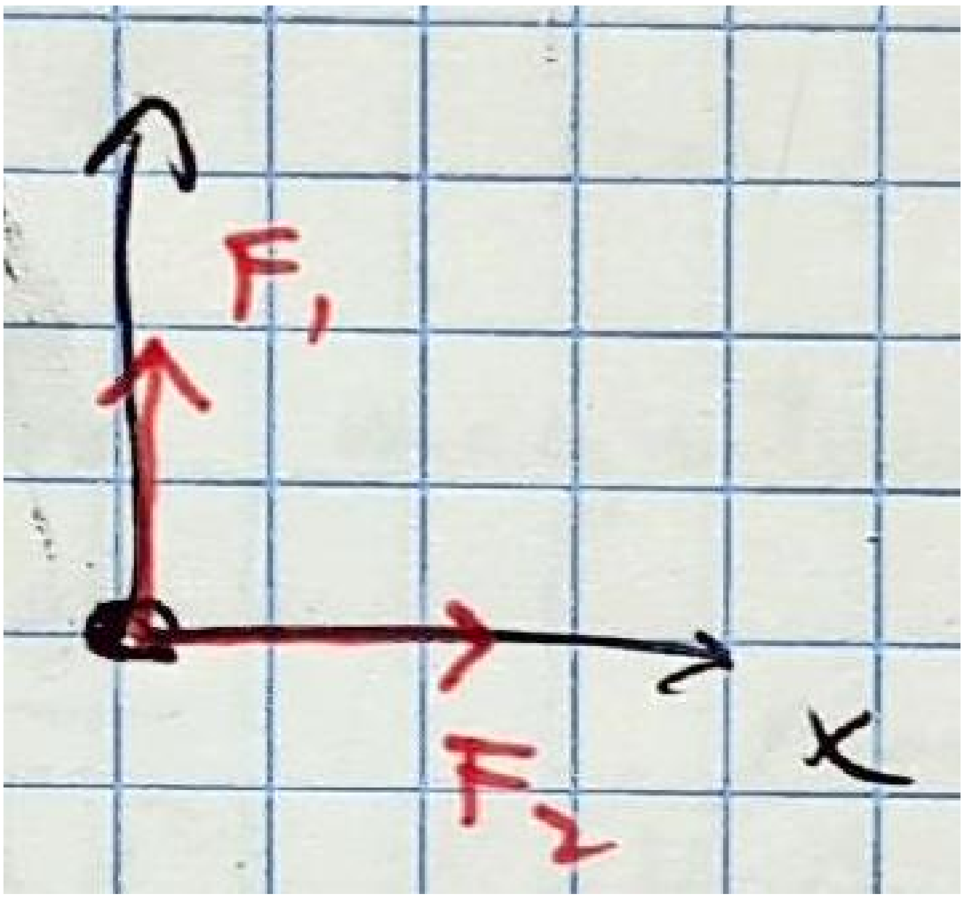
Choose x, y coordinates such that the two forces are aligned with the axes.

Since the horse is in deep space we can assume that there are no other forces on the horses.

Specifically, we cannot consider the horse's weight in this scenario. We ONLY have weight near the surface of some planet or star or really any massive object. Here there is nothing around to impart weight. We still assuredly have mass: that goes with us everywhere no matter what!

$$\vec{w} = 0$$

Let's wake the horse a single dot at the origin and sketch our forces:



The key to this problem is to add the two vectors as Newton reminds us to do.

$$\begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} + \begin{bmatrix} F_{2x} \\ F_{2y} \end{bmatrix} \quad = m \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

Plugging in our values

$$\begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} + \begin{bmatrix} F_{2x} \\ F_{2y} \end{bmatrix} \quad = \begin{bmatrix} 0 \\ 100 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \end{bmatrix} \quad = \begin{bmatrix} 50 \\ 100 \end{bmatrix} = 1000 \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

So then, solving for each component of acceleration we have

$$\vec{a} = \begin{bmatrix} 50/1000 \\ 100/1000 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}$$

Now, let's find the magnitude and the direction: these are cartesian components we acquired but it doesn't really give us a good intuition or visualization where the horse is heading. What we'd like are the polar components of the acceleration.

$$|\vec{a}| = \sqrt{0.05^2 + 0.1^2} = \sqrt{0.0025 + 0.01}$$

$$|a| = 0.11 \ldots m/s^2$$

Now, the direction is given by

$$\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{100}{50}\right) = 63.4°$$

We can handle ANY forces this way so long as we know what they are. Usually we only have access to a "model" of what the force is.

Let's consider friction.

Friction is a force which opposes the motion of an object. The model we shall use for friction is

$$f_{\mu s} = \mu|\vec{N}|$$

This is an approximation. friction is enormously complex and involves many different aspects of the total system under study. The model we use is the simplest. This says that the frictional force is due entirely to the Normal force and a number μ (the coefficient of friction) which encodes all of the specific details about the two surfaces. μ is typically less than unity, however you might suspect that glue would have an extremely large value for μ. We wont go into any details on why this model is apt, we will instead consider it to be good enough and then compare it to experiments.

Notice that what we've written here is the magnitude of the frictional force. The direction is found by considering which way the object is trying to move. Friction is a force so to find its direction you need to consider which way your expecting your object to move.

Now friction is further broken down into two main types: static and kinetic friction.

Static friction, called $f_{\mu s}$, is the force needed to get an object to accelerate during the time it is not accelerating. We will take this to be

$$f_{\mu s} = \mu_s |N|$$

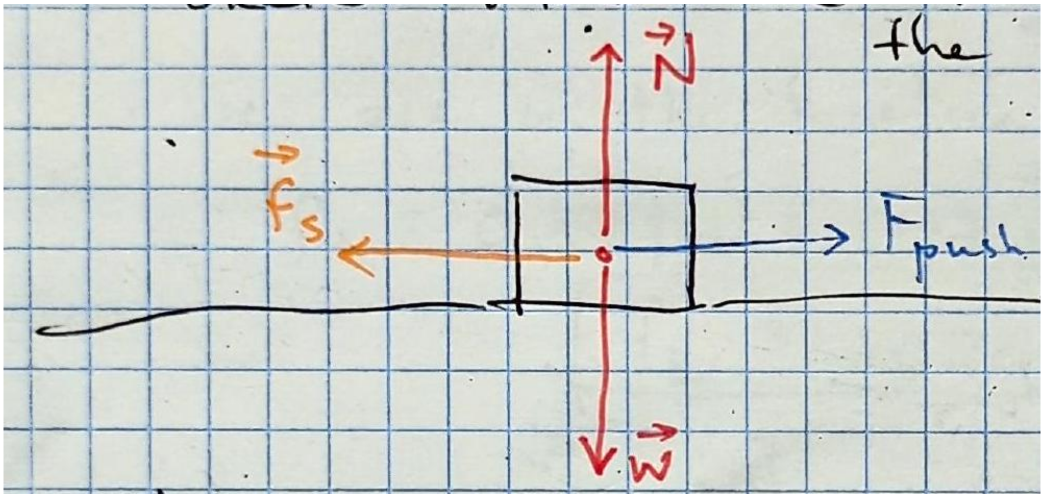
Kinetic friction is the friction that we experience when the object is moving. It is typically less than or equal to $f_{\mu s}$

$$f_{\mu k} = \mu_k |\vec{N}| \leq f_{\mu s}$$

Let's see how all this comes together in a practical problem.

Q1: You push a box across the floor using a forklift where $m_{box} =$ 600kg. What force must the forklift apply just to get it moving? Let $f_S = 0.28|\vec{N}| \quad f_k = 0.17|\vec{N}|$

Same steps as always: Sketch FBD and choose coordinates. We will have four total forces acting on our box as below:



Here we explicitly consider static friction since the box hasnt moved yet. Let us write Newton's second law and then fill in what we know

$$\sum \vec{F}_i = \begin{bmatrix} F_x \\ F_y \end{bmatrix} + \begin{bmatrix} N_x \\ N_y \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \end{bmatrix} + \begin{bmatrix} f_{\mu x} \\ f_{\mu y} \end{bmatrix} = m \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

Now then, the normal and the weight vectors are entirely in the y-direction and the push and friction forces are entirely in the x-direction. So we fill in,

$$\sum \vec{F}_i = \begin{bmatrix} F_{push} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ N \end{bmatrix} + \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} -0.28N \\ 0 \end{bmatrix} = m \begin{bmatrix} a_x \\ 0 \end{bmatrix}$$

Noticed that we set $a_y = 0$ since we assuredly do not expect our box to float away or fall through the earth! Additionally, we went ahead and plugged in our static friction model. Note especially the explicit minus sign: we MUST put that in by hand. That is the direction of the force.

So, we find we have an equation for the x direction and one for the y direction.

$$\sum \vec{F}_i = \begin{bmatrix} F_{push} - 0.28N \\ N - mg \end{bmatrix} = m \begin{bmatrix} a_x \\ 0 \end{bmatrix}$$

We have two equations with three unknown quantities: a_x F_{push}, and N. We see that we may solve N directly

$$N = mg = 600 \times 10 = 6000$$

$N = 6000$ Newtons.

Now we can plug this into our x-direction equation:

$$F_{push} - 0.28N = ma_x = F_{push} - 1680 = 600a_x$$

Now we're stuck. We need to either know how hard we push, or the acceleration. You might sit and dwell on this conundrum for a day or two but eventually you will come to the realization that during the static friction phase the box is not moving. If we can find the maximum possible force during this no-motion phase then we could argue that any teeny-tiny little bit of force more will cause a non-zero a_x

So, our strategy is instead to take $a_x = 0$. Now we can find out how hard we must push because any infinitessimally small force above this value will cause acceleration however small that may be.

so,

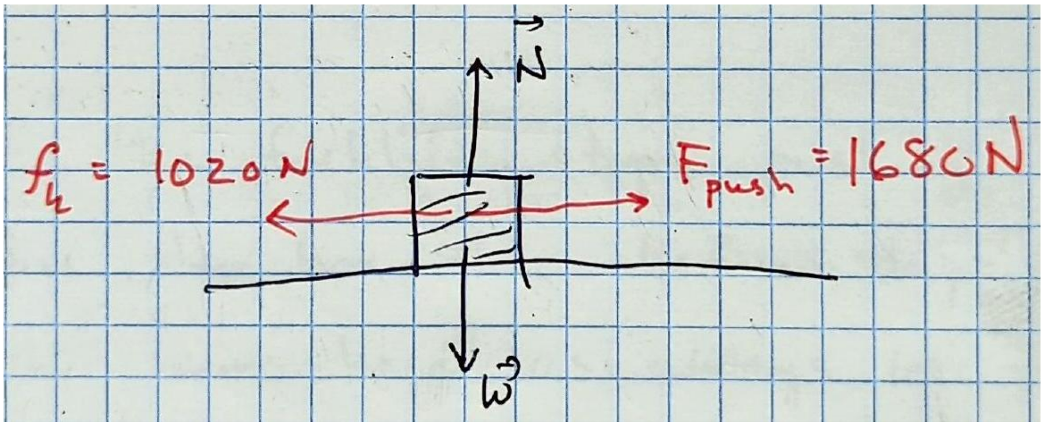
$$F_{push} - 0.28N = 0 \Rightarrow F_{push} = 1680$$

Just a tiny bit more than 1680 Newtons, say, 1680.0000001N will induce an acceleration. In other words we must overcome friction in order to push the box. The question is solved

Q2: The pallet now begins to slide. How fast is the box moving if the push is still 1680 N after 0.5 seconds.

This question seems tricky. Let's think:

The box is now moving and therefore we are in the kinetic friction regime. In this case the whole problem acts like a brand new system insofar as forces are concerned. Let's go all the way back to step 1: re-draw , label, and FBD:



Here since we know N (that doesn't change) we may plug this into our model for kinetic friction. Viz,

$$f_k = -0.17|N| = -0.17 \times 6000$$
$$f_k = -1020$$

Our friction is 1020N; considerably less than the static friction.

Notice now that the forces in the y-direction cancel since we still do not expect any acceleration in our vertical direction.

But there is a net force along x because when you add the x components of the forces in this direction you get

$$F_{push} > f_k$$

and so

$$F_{total} = 1680 - 1020 = 660$$

but $\vec{F_t} = m\vec{a} \Rightarrow a = \frac{660}{600} = 1.1$ m/s^2. The box is accelerating!

Since it accelerates for $\frac{1}{2}$ second then we can call Galileo to find velocity

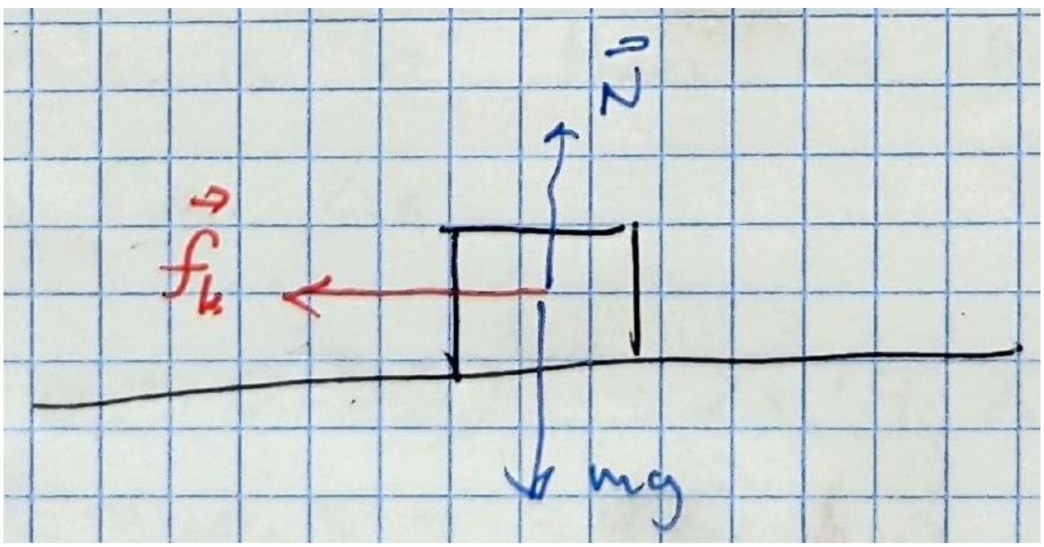
$$v_f = v_i + at$$
$$v_i = 0 \quad v_f = 1.1 \times \frac{1}{2} = 0.55 \text{ m/s}$$

Hooray! To solve this difficult problem we simply followed our noses.

The KEY is to be careful about minus signs!

Q3: If the forklift stops pushing at this instant then how far does the box slide before stopping?

Yikes! This question seems much harder! We don't have a lot of information to go off of. Let's follow our approach and, as always, sketch and draw the FBD



We realize that the only force on the box in the x-direction is $\vec{f_{\mu k}}$. The y-direction is entirely the same, $N = 6000$N. We need only consider the x-direction, but here there's just a single force!

$$F_{total} = -f_{\mu k} = ma_x$$

but $f_{\mu k} = 1020$ Newtons

So that $a_x = -1020/600 = -1.7m/s^2$

But Galileo tells us that if we know the position, velocity, and acceleration then we know what the object does!

For this problem

$$v_i = 0.55m/s$$

and we require the box comes to a stop, so

$$v_f = 0$$

since we know both v_i and v_f then

$$v_f^2 = v_i^2 + 2a\Delta x$$

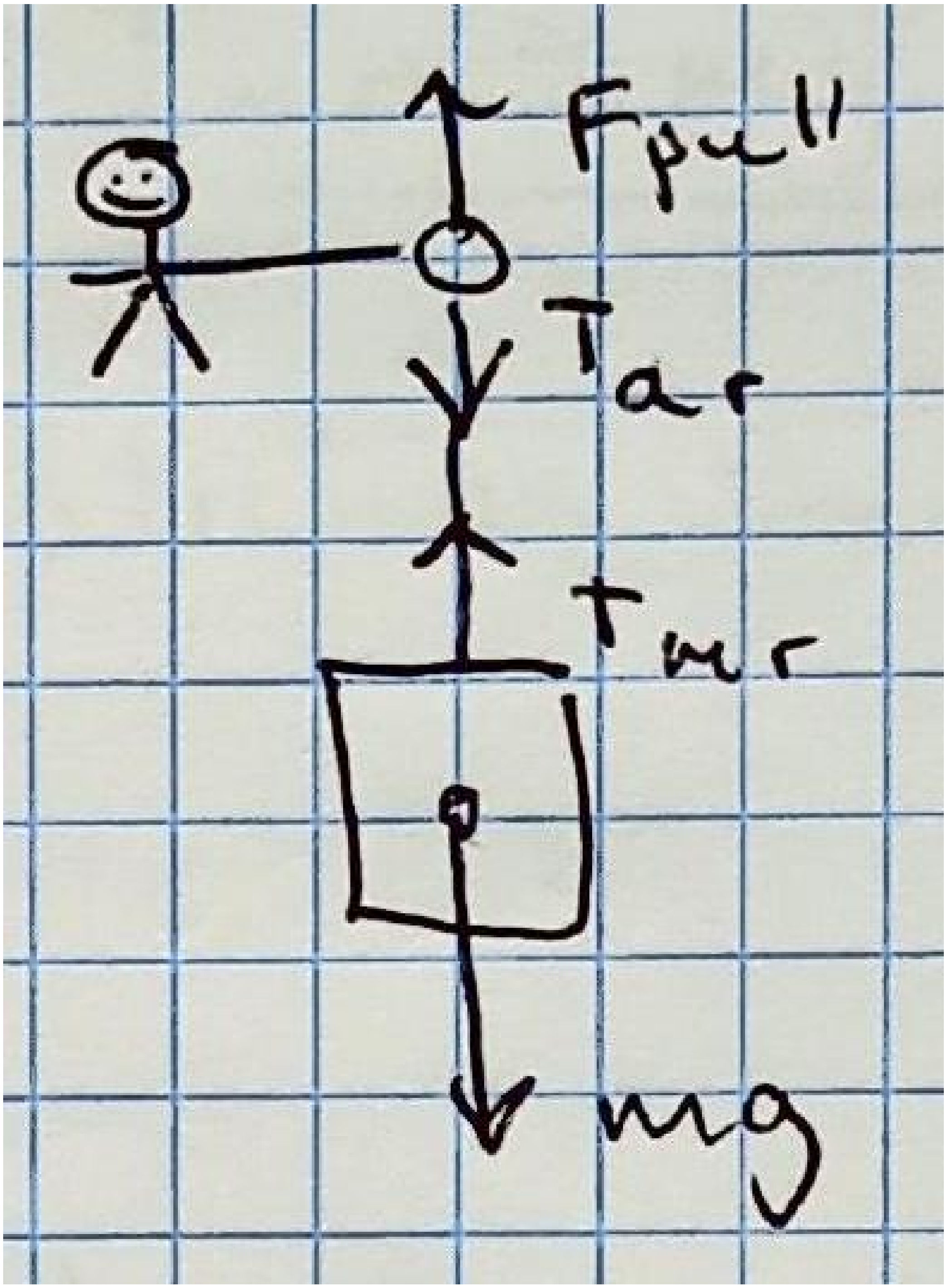
will allows us to solve for Δx - the distance the box slides. Plugging in, we find,

$$\begin{aligned}
&v_f^2 = 0 = (0.55)^2 - 2 \times 1.7\Delta x \\
&= 0.3 - 3.4\Delta x = 0 \\
&\Rightarrow \Delta x = 0.09 \text{ m}
\end{aligned}$$

If you could understand the logic in all of these questions then you'll be able to solve most of the problems you face.

The next force we will discuss is Tension. Tension is a force along a length of material. Usually we think of a rope or rubber band or cable. These things allow us to direct a force along different directions (i.e. it transmits force from one place to another).

Suppose we lift a weight by pulling on a rope with some force i call F_{pull}. Then our FBD looks like


Fpull
Tar
Tmr
mg

We have here many forces but notice that if $T_{mr} \neq T_{ar}$ then there must be a net acceleration of the system. But that doesnt happen in reality! So therefore we should say $T_{mr} = T_{ar} = T$.

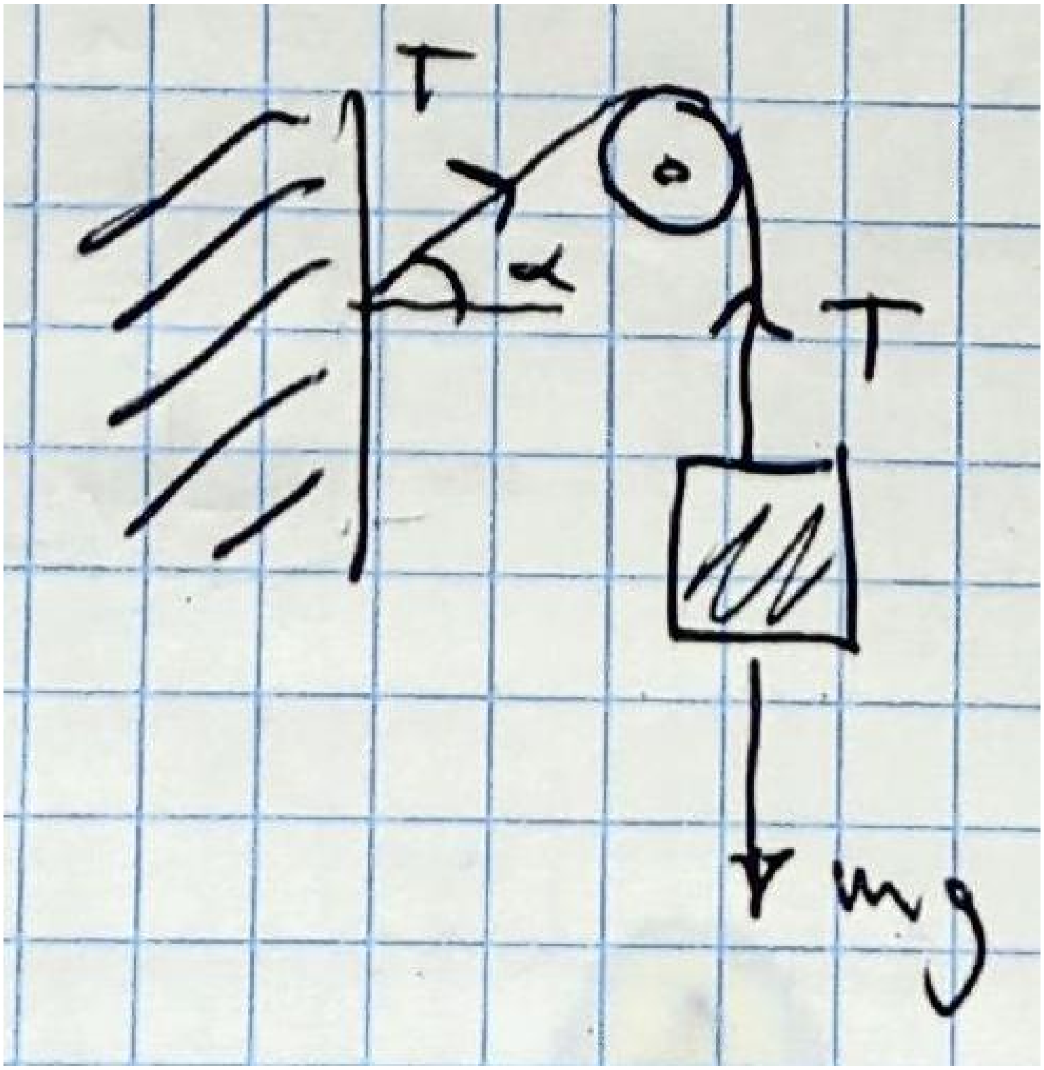
They are equal and opposite so that they produce no acceleration.

Next we have w and F_{pull}. If im just holding the weight then that implies there is no net acceleration between the weight and my pull. Therefore, $mg = F_{pull}$. It's entriely equivalent to holding the weight itself, rather than holding it via the rope.

In total we would solve $\vec{F}_{pull} + \vec{T}_{ar} + \vec{T}_{mr} + \vec{w} = 0 = m\vec{a}$

Importantly: typically we consider the rope/cable itself to have zero mass and is unable to stretch. This is a massive simplification and a more accurate treatment would involve considering those other forces in the analysis.

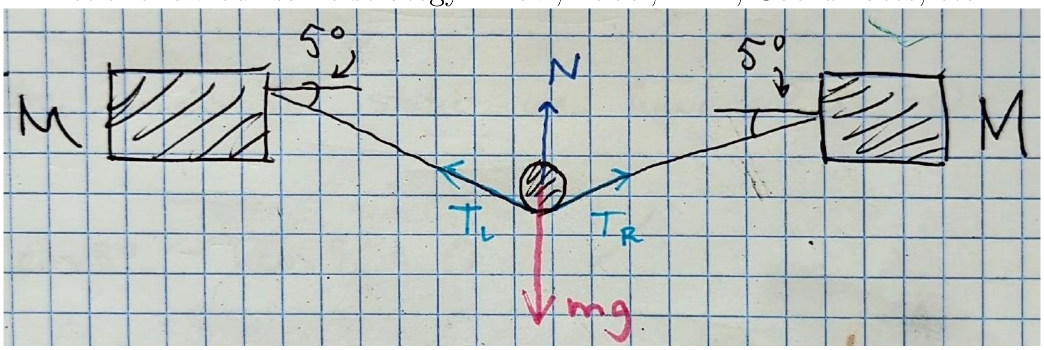
The beautiful thing about tension is that it allows you to transmit a force around curves or between distinct places without any difficulty. For example, see the figure below



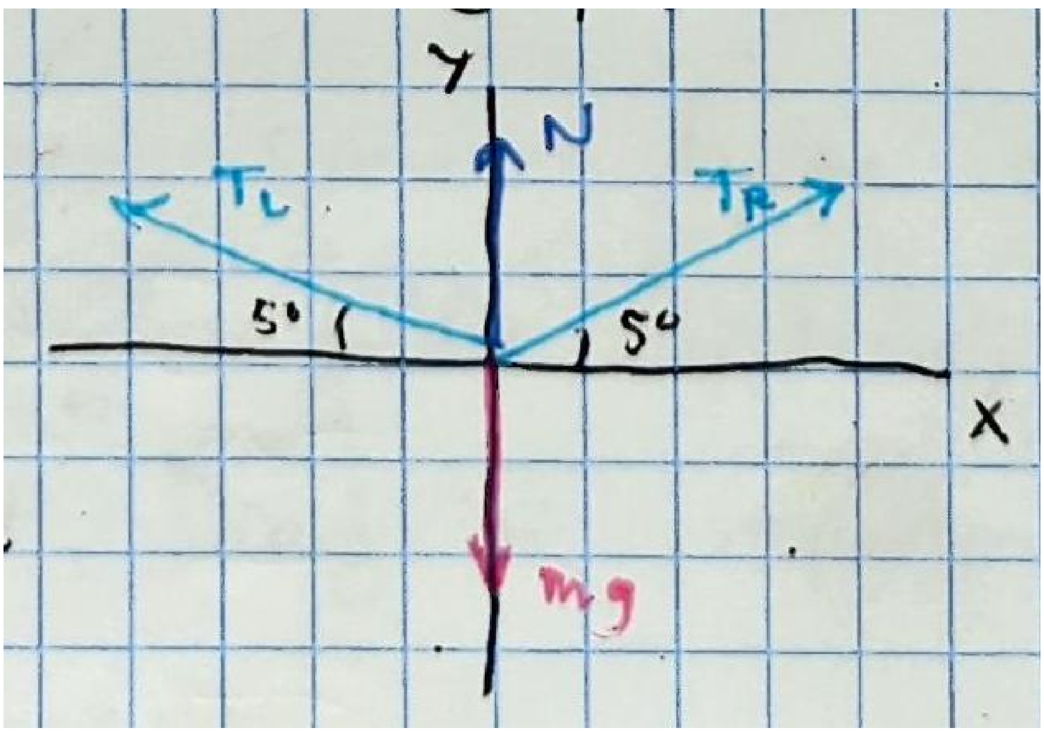
Tight Rope

Suppose a person stands in the middle of a tight rope and the rope is connected to two massive blocks/buildings such that under load the rope deflects by $5°$ as in the figure. Find the tension in the rope.

Let's follow our same strategy. Draw, Label, FBD, Coordinates, etc



Here we have a normal force (a surface!) but it is not being provided by the ground. If we add up all the forces pointing up then these should provide the normal force. Let us choose coordinates with the contact point (the guy) to be $(0, 0)$



The normal force is produced by the tension. In fact we should NOT draw the normal force in this case because we know where that force is coming from. In the figures i draw it only for clarity, but this is NOT a force we will include in Newton's Laws. If we do, we'll find it to be counted twice. So we only consider weight and tension: no normal

That being said, we must now decompose all of our vectors and then use $\sum F_i = ma$.

The weight vector is easy. It is entirely in the $-y$-direction with magnitude mg. The tension is a little trickier, but not much. Consult the FBD with coordinates and go slowly.

$$\vec{T_R} = \begin{pmatrix} T_{Rx} \\ T_{Ry} \end{pmatrix} = \begin{bmatrix} T\cos 5^\circ \\ T\sin 5^\circ \end{bmatrix}$$
$$\vec{T_L} = \begin{pmatrix} T_{Lx} \\ T_{Ly} \end{pmatrix} = \begin{bmatrix} -T\cos 5^\circ \\ T\sin 5^\circ \end{bmatrix}$$

and $\vec{N} = \vec{T_L} + \vec{T_R}$

Since the mass is not accelerating (WE HOPE!!) we then have $\sum F = 0$

$\underline{x - direction}$

$$T\cos 5^\circ - T\cos 5^\circ = 0$$

$\underline{y - direction}$

$$T\sin 5^\circ + T\sin 5^\circ - mg = 0$$
$$\Rightarrow T = \frac{mg}{2\sin 5^\circ}\Bigg|$$

Drag and "Wind Resistance"

Drag is another common type of force. Like friction, if opposes motion. The magnitude of drag is proportional to the square of the speed. This makes solving problems involving drag really challenging. Typically, under forces an object will accelerate meaning that its speed is constantly changing over time. This means the drag force is constantly changing. It is what is known as a continuous vector valued function and most of the time we need methods from calculus to solve drag problems.

We shall consider the model for drag as

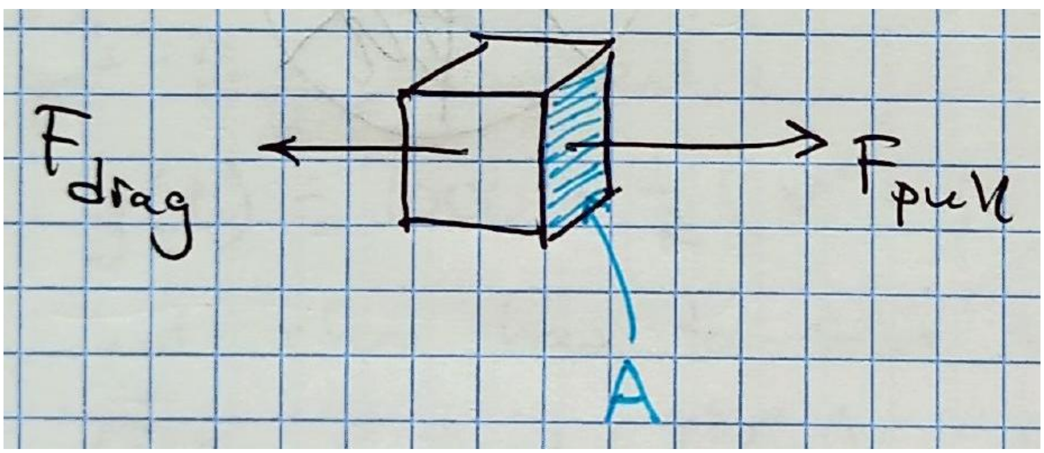
$$|F_D| = \kappa v^2$$

κ is a number that depends not only on the object but also on the medium through which the object moves.

in particular

$$\kappa = \frac{1}{2}CA\rho$$

where C is the "drag coefficient", A is the cross-sectional area of the object perpendicular to its motion, and ρ is the density of the material the object is moving through. See below



For example, for a skydiver $C = 1.0$ and for an airplane wing $C = 0.05$

Let's see a problem we can solve:

Suppose we drop a ball towards the earth in the presence of air. We will allow the object to fall for an indefinite amount of time. As the ball accelerates and picks up speed the drag force increases quadratically. At some point, we expect the drag force will approach the weight of the object itself and the acceleration will decrease until the drag perfectly balances the weight. This will be equilibrium and the acceleration will be 0 since the forces are equal and opposite.

Therefore,

$$\sum F_x = 0$$
$$\sum F_y = F_D - mg = 0$$

but $F_D = \kappa v^2$ so that

$$v_T = \sqrt{\frac{gm}{\kappa}}$$

This is known as the terminal velocity. It is the fastest a falling object can move under drag. Parachutes have a MUCH larger κ than people. They are able to slow down a LOT due to this effect.

Now, if we wanted the velocity at any time, not just in equilibrium then we could proceed the same way, only now we do not assume $a_y = 0$

$$\sum F_y = F_D - mg = a_y$$

$$\Rightarrow a_y = \frac{\kappa v^2 - g}{m}$$

This is called a "differential equation" and can be solved using calculus. Its solution for $v(t)$ is

$$v(t) = v_T \tanh\left(gt/v_T\right)$$

where V_T is the terminal velocity and $\tanh(x)$ is called the hyperbolic tangent function. Try to plot this equation for $v(t)$ by picking some simple values for mass and κ.

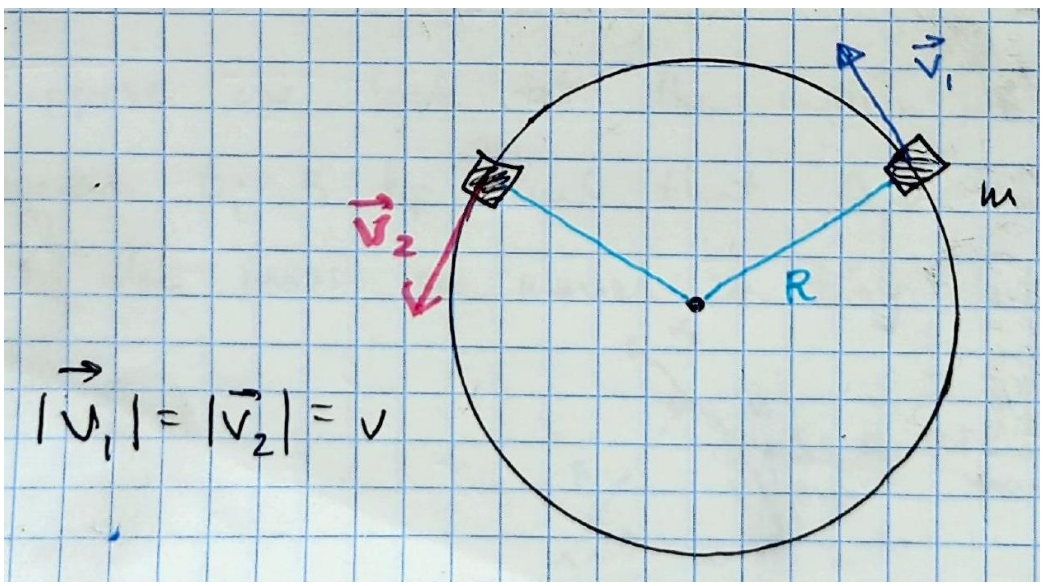
Uniform Circular Motion

Suppose we wish to understand the motion of a mass attached to a thin string moving along a circular path around our head. These types of motions occur all throughout the universe be they rollercoasters, planets, or atoms.

We shall not consider the most general curved paths only the special case of "uniform" and "circular" motion. The full treatment requires some calculus but the principles studied there are identical to those studied here.

The circular requirement is pretty obvious : the motion is along a circle.

By "uniform' we mean that the "speed" along this path is constant. This does NOT mean the "velocity" is constant. Remember, the velocity is a vector and there are two ways to change a vector: you can change the length or you can change its direction. In our UC case we demand the length of the velocity vector stays constant but its direction is certainly always changing as the mass goes along its circular path. See figure below



Recall Newton's first law: an object travels at constant velocity if there are no forces acting.

Similarly recall Newtons second Law

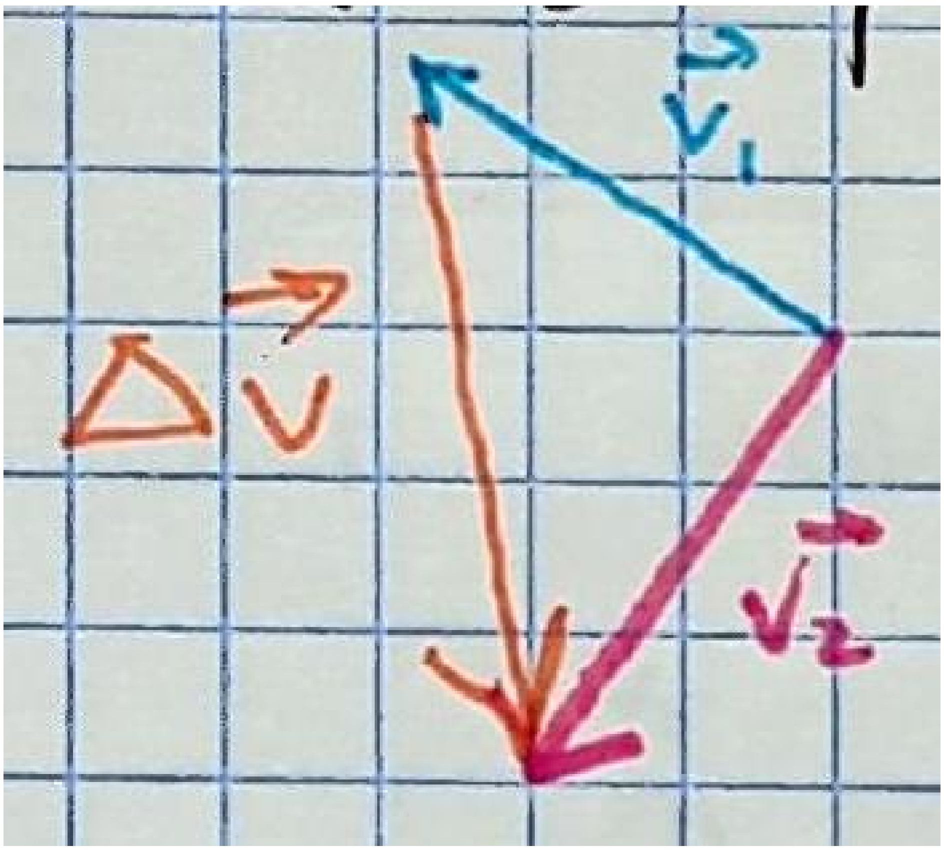
$$\sum F = m\frac{\Delta \vec{v}}{\Delta t}$$

In our UC case the direction of the velocity vector changes which implies

$$\Delta \vec{v} \neq 0$$

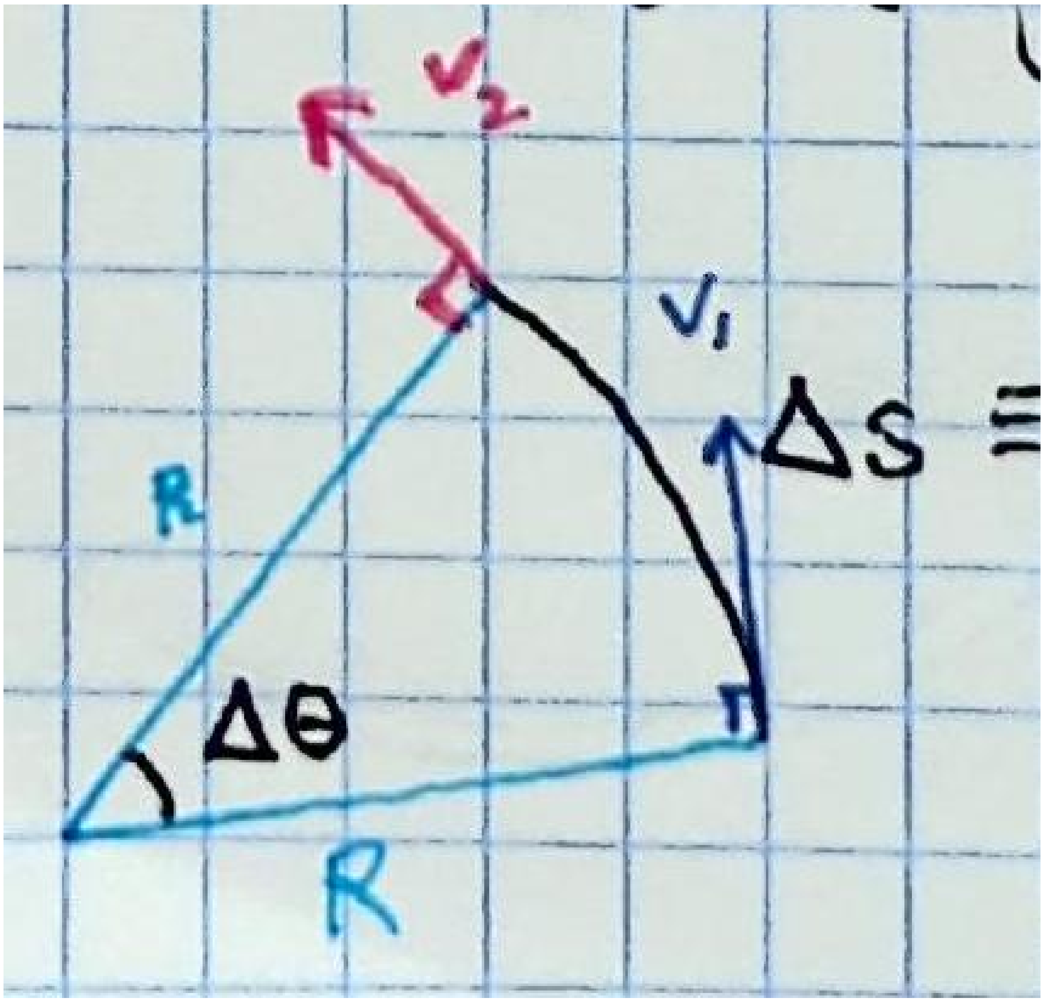
Yet we also have $\Delta|\vec{v}| = 0$. This feels like a contradiction, however, a vector can change its direction without changing its length. Prove this.

So then, consider the two points in the figure, then between point 1 and point 2 $\quad \Delta\vec{v} = \vec{v}_2 - \vec{v}_1$



Since $\Delta\vec{v} \neq 0$ then this implies via Newton that there exists a constant Force perpetually changing the direction of the velocity vector. Let's follow our noses to try and figure out what mysterious force this might be.

Suppose we look at the motion of the mass between times t_i and t_f such that $\Delta t \ll 1$. In this limit the mass moves a tiny incremental bit along an arc. In principle we could make this arc as small as possible by choose more and more accurate stopwatches. See below



Let Δs be the arclength between these times. This is the path the mass move along. If that's the case then we can figure out its speed by the definition of speed: distance over time.

Our plan is to say that since the speed of the mass is constant and the distance the object moves is Δs then the speed will be

$$|v| = \frac{\Delta s}{\Delta t}$$

but, by definition of a circular arc length

$$\Delta s = R\Delta\theta$$

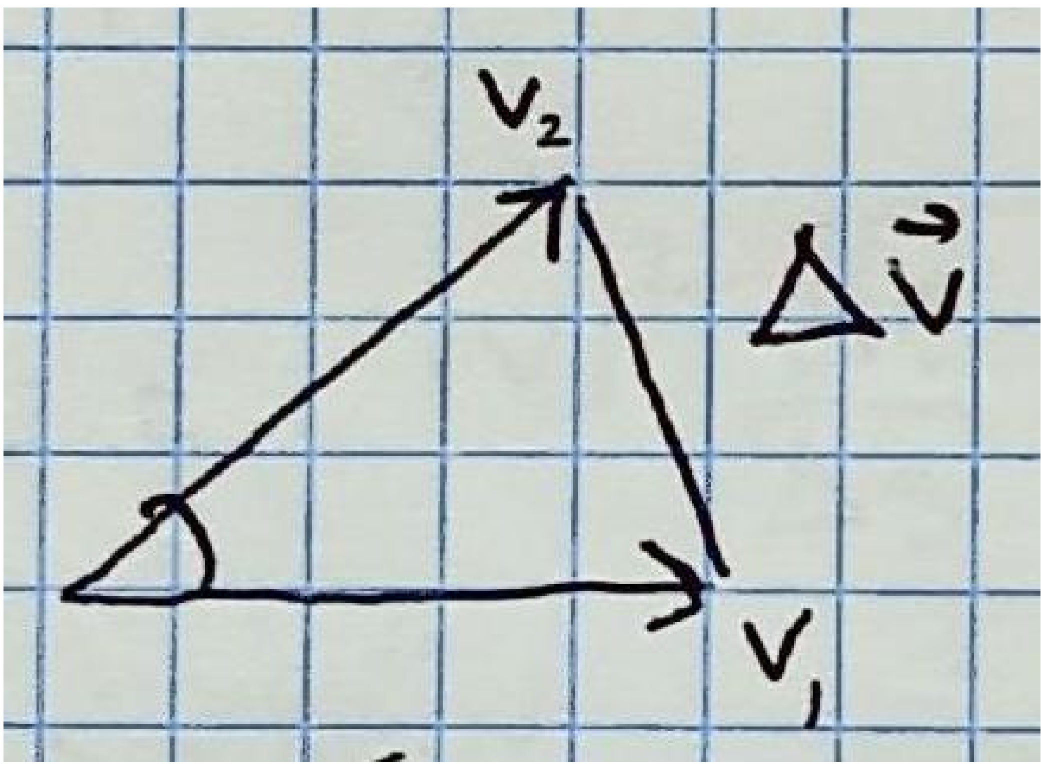
so that

$$|v| = \frac{R\Delta\theta}{\Delta t}$$

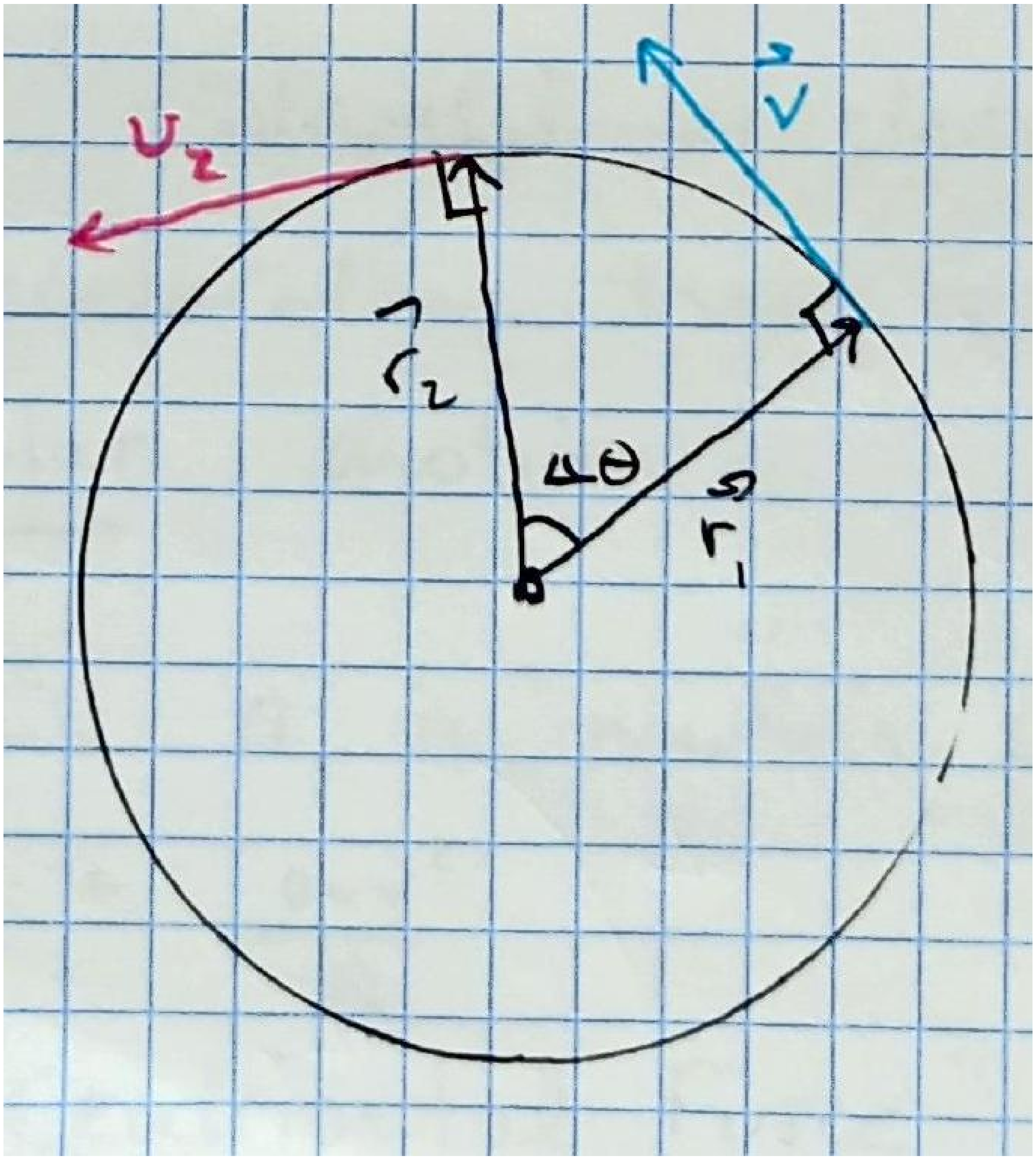
Now, ultimately we are looking for an acceleration

$$a = \frac{\Delta\vec{v}}{\Delta t}$$

Therefore, let's look at our velocity triangle; this gives us $\Delta\vec{v}$ as a vector.



Here since the mass is moving in a circle this implies that $\vec{v} \perp \vec{r}$ where $\vec{r}$ is the position vector of the mass. Then, by Euclid, we can also find $\Delta\vec{r}$



When we do this, since the angles are all the same we see that the position and velocity triangles are similar!

Since $\Delta t \ll 1$ then surely $\Delta\theta \ll 1$ and so

$$\Delta\vec{r} \approx \Delta s = R\Delta\theta$$

and

$$\Delta v \approx |v|\Delta\theta$$

But recall we found $\frac{\Delta s}{\Delta t} = |v|$ so that we can spolve this for Δt

$$\Delta t = \frac{\Delta s}{|v|}$$

Finally, to find $a = \frac{\Delta v}{\Delta t}$ we plug inwhat we've found

$$a = \frac{|v|\Delta\theta}{\Delta s/|v|} = |v|^2\frac{\Delta\theta}{\Delta s}$$

But $\Delta s = R\Delta\theta$, then, finally

$$a = \frac{|v|^2\Delta\theta}{R\Delta\theta} = \frac{v^2}{R}$$

And so we have found the acceleration that must occur for uniform circular motion!. We call this "centripetal acceleration" a_c.

We can use this a_c in Newton's Laws and find that for UC motion there MUST exist some force radially oriented towards the center of the circle such that

$$F_c = ma_c = \frac{mv^2}{R}$$

This is called a "Centripetal Force" meaning "center seeking". It is ALWAYS directed into the center of curvature - i.e. towards the center of the circular arc.

Now, this is a quite subtle and difficult point. We do not know what F_c is in the sense of Free Body Diagrams much like we don't know what the normal force is. It could be anything that constrains the motion to lie on a circle. In the case of the mass on a string it is the Tension in the string that is supplying the centripetal force. For the earth going around the sun then it is Gravity itself that causes the velocity vector to swing around continuously changing directions (the earth doesn't exactly follow a circle so this is approximate).

When we solve a UC motion problem we take our regular FBD and then define one of our coordinate axes to lie along the radius of the circle and then ALL of our radial forces add together to produce a centripetal force directed towards the center. There could be MANY forces all conspiring to produce a centripetal force.

For example, the force due to gravity is modeled as $F_G = -G\frac{m_E m_S}{r^2}$ is the force of gravity due to the sun on the Earth. If the planet is in uniform circular motion then we write

$$F_G = F_C = m_E a_c = \frac{m_E v^2}{r}$$

Then, plugging in

$$G\frac{m_E m_S}{R_E^2} = \frac{m_E v_E^2}{R_E} \Rightarrow v_E^2 = \frac{Gm_s}{R_E}$$

this is a famous prediction that finds the speed of the planets around the sun are given by

$$v_0 = \sqrt{\frac{Gm_s}{R_0}}$$

where R_0 is the distance the planet is from the sun.

Often we will compare a_c to g.

For example, suppose you are an astronaut training and you hop into a centrifuge with radius 10 meters then you will be accelerated centripetally via

$$a_c = \frac{v^2}{R}$$

This centripetal acceleration is provided by the normal force between you and the capsule. Suppose the goal is to experience $10g$'s. Then the operators can calculate what constant speed the capsule must move

$$\begin{aligned} 10g &= 100 \text{ m/s}^2 = \frac{v^2}{10} \\ \Rightarrow& v^2 = 1000\frac{\text{m}^2}{\text{s}^2} \\ \Rightarrow& |v = 31.6 \text{ m/s} \end{aligned}$$

Usually we do not know $|v| = \frac{\Delta s}{\Delta t}$ but instead know the time it takes to execute one rotation. We call this the "Period" T. Equivalently, we can count how many tines the object goes around each second. This is called the frequency f. Frequency has units of s^{-1} called Hertz (Hz).

Now, if we take the total angle (in radians) then for 1 revolution ($= 2\pi$ radians) and divide by the period we have what is called the angular velocity ω (pronounced omega).

$$\omega = \frac{2\pi}{T} = \frac{\Delta\theta}{\Delta t}$$

This is how fast the angle is changing (in radians per second).

Now for 1 full revolution then the mass traverses a total distance of $2\pi R$ (the circumference!). The velocity, v, will be

$$\frac{2\pi R}{T} = v$$

but

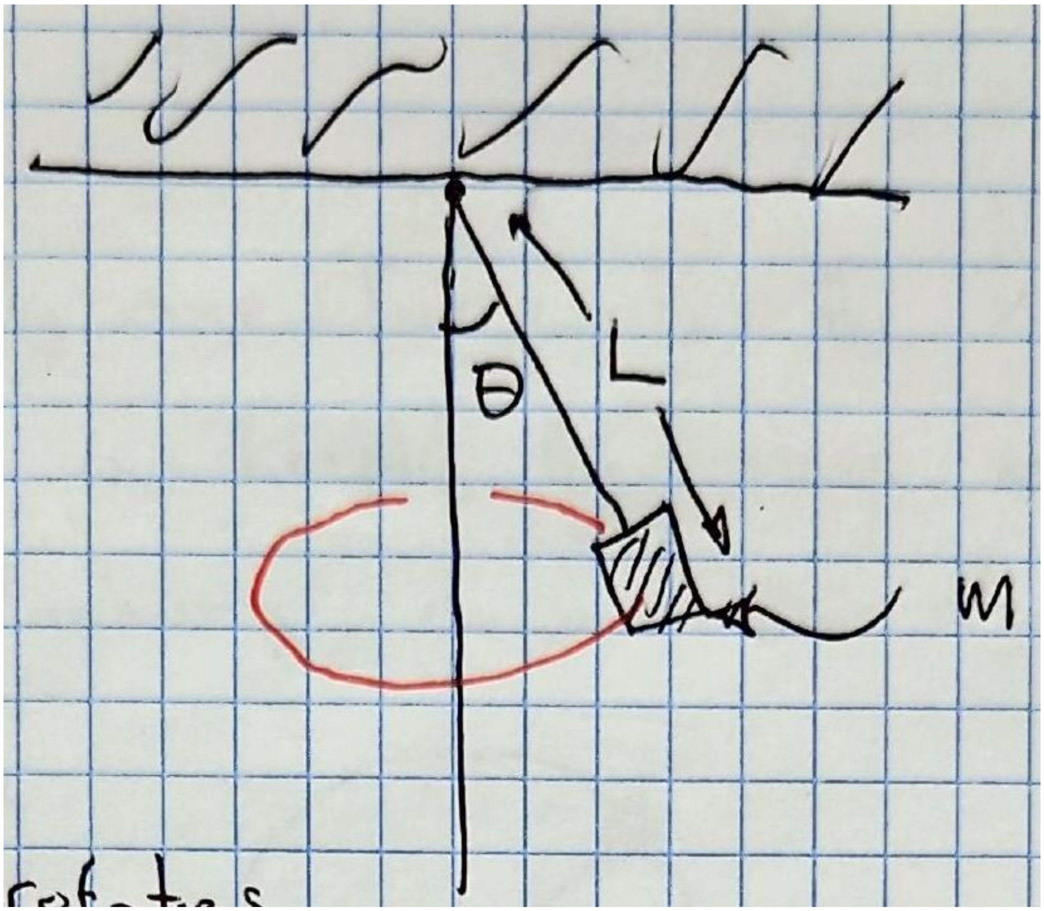
$$\omega = \frac{2\pi}{T} \Rightarrow v = \omega R$$

So we can instead write a_c as

$$a_c = \omega^2 R$$

This form is generally much handier than the one involving linear speeds. The details of a particular problem will often make clear which version is preferable. They are identical but one is in angular coordinates and the other is in cartesian coordinates.

Example - A Conical Pendulum



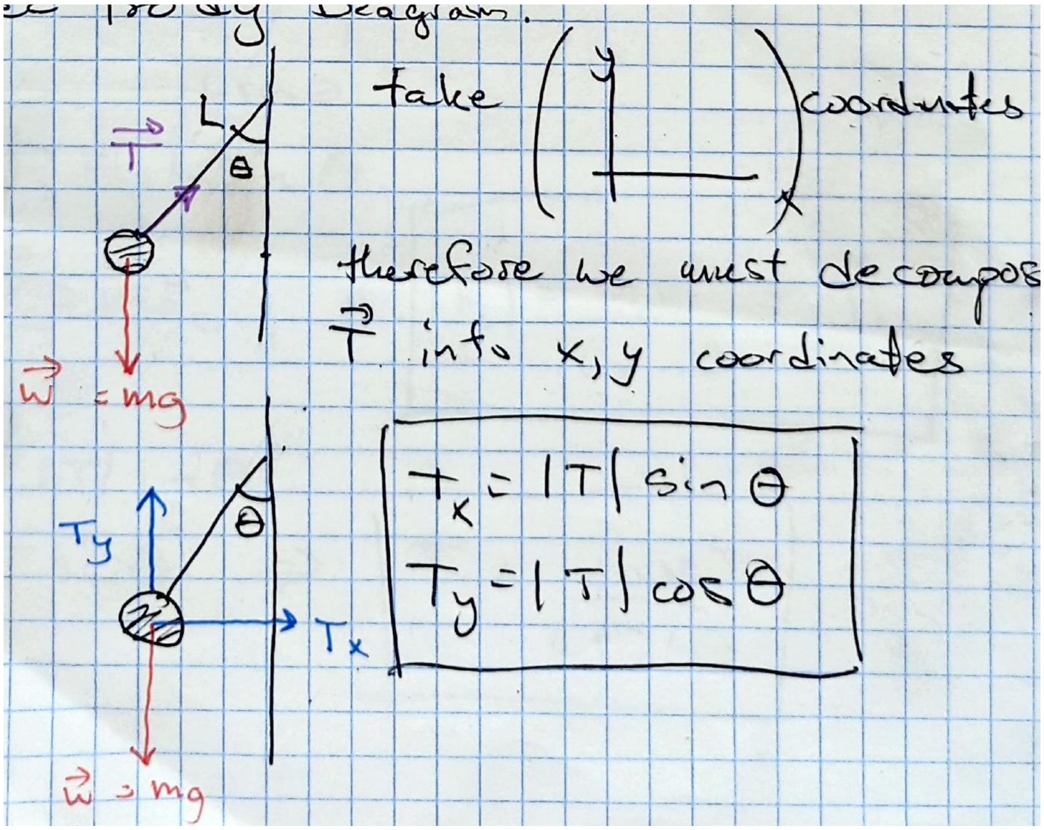
A pendulum rotates about the y-axis at an angle θ with period τ (we call it τ instead of T because we also have a tension in this problem).

We ask:

a) What is the centripetal force

b) What speed does it rotate with?

Step 1: as always we sketch and draw the Free Body Diagram!

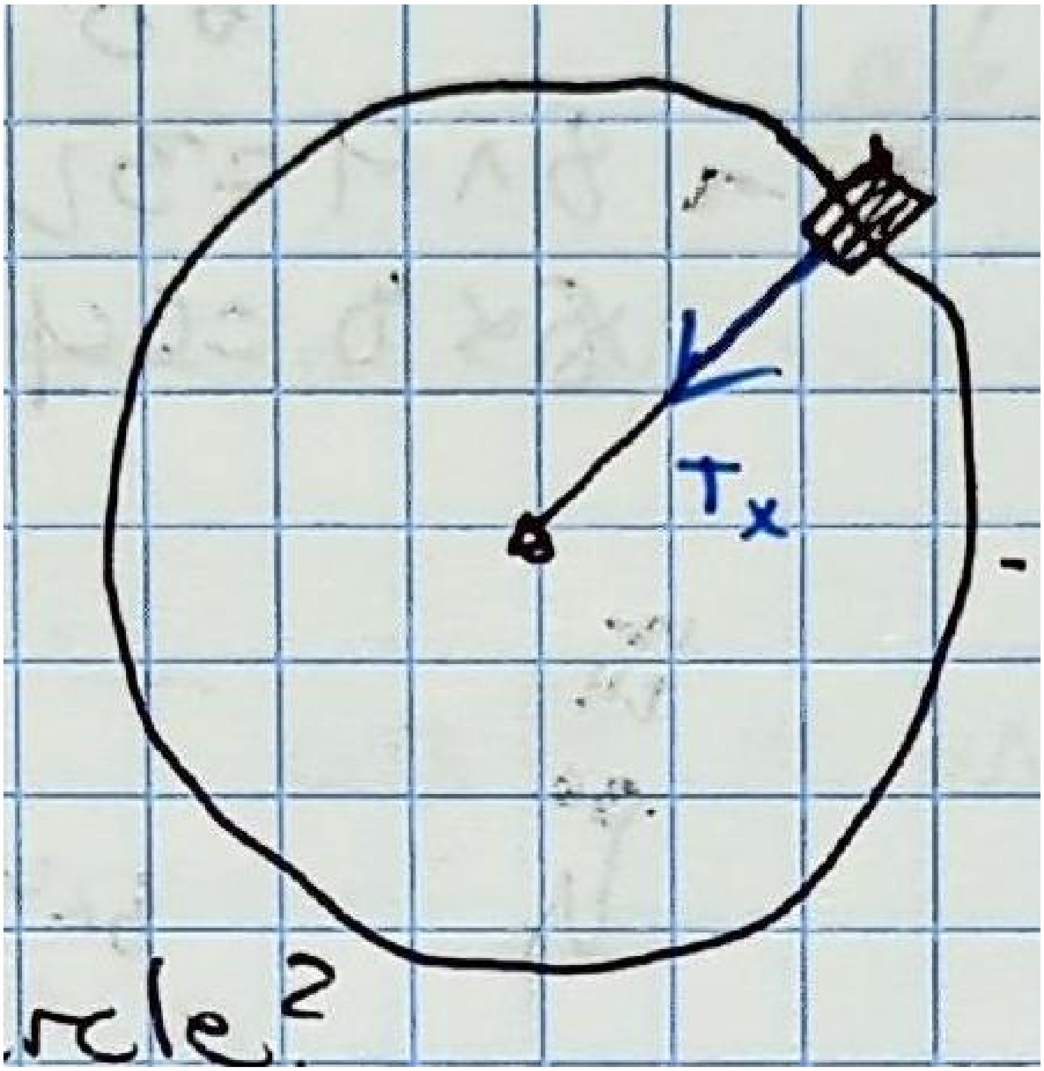


We take the coordinates implied by our questions where the y-axis lines up with the line of symmetry of the rotation. The x-axis will lie along the radius of the orbit.

Therefore we must decompose $\vec{T}$ and $\vec{w}$ into x, y coordinates. From the diagram we have

$$T_x = |T| \sin\theta$$
$$T_y = |T| \cos\theta$$

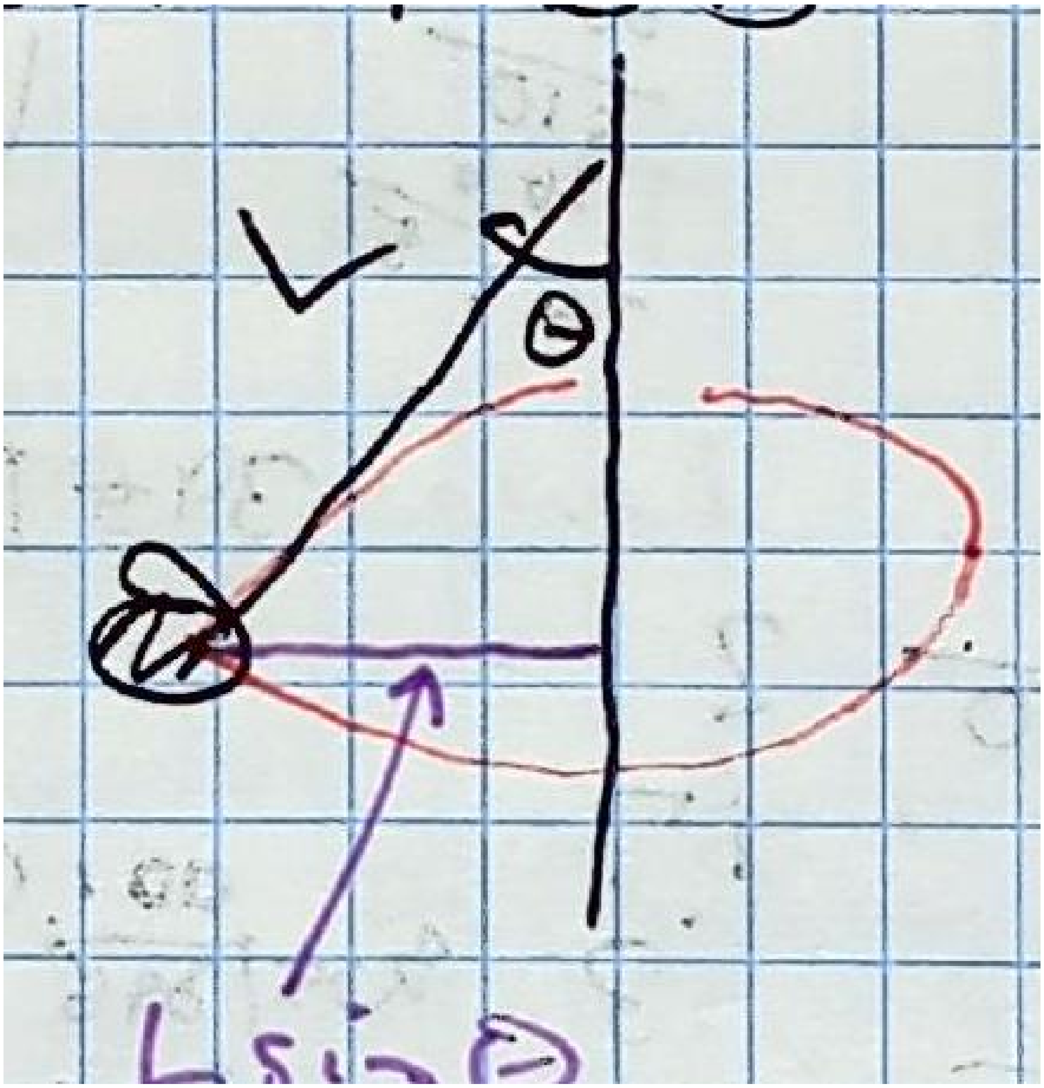
There is only one force (Tension) in the x-direction and this is keeping the mass moving in a circle. Looking down from above we would see the following picture



Therefore the only radial force is the component of tension in the x-direction.

$$\vec{T}_x = T \sin\theta = \vec{F}_c = \frac{mv^2}{R}$$

But what is the radius of the circle? Let's look at our FBD.



The mass goes around the y-axis. The radius is NOT the length of rope, but rather $L\sin\theta$ according to the geometry. So then, plugging in

$$T\sin\theta = \frac{mv^2}{L\sin\theta}$$

so that

$$T = \frac{mv^2}{L\sin^2\theta}$$

This is all well and good but how do we find velocity?

We solved Newtons 2^{nd} Law in the x-direction. In the y-direction we expect $a_y = 0$ if θ does not increase/decrease. Then

$$\sum F_y = 0 = T\cos\theta - mg$$

$$T = \frac{mg}{\cos\theta}$$

Now we may plug this into our expression and find

$$v^2 = Lg\tan\theta\sin\theta$$

Example - Loop d' Loop

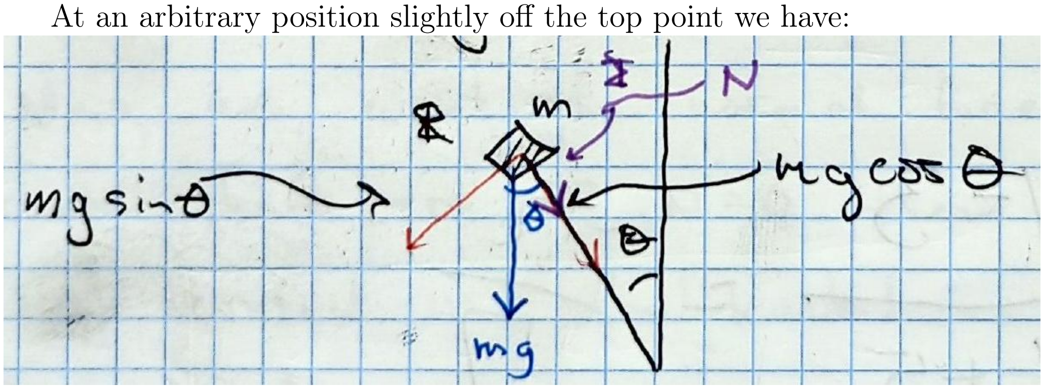
Find the minimum speed needed for a roller coaster to make a loop of Radius R

Step 1, as always, sketch, FBD, coordinates

Let's analyze this when the cart is at the very top. If we can find a condition on the speed to get to this point then maybe we can argue that any speed greater than that will get us around.

At the top we have two forces acting on the cart. The normal force and the weight. The normal and the weight BOTH point down towards the ground

At an arbitrary position slightly off the top point we have:



The centripetal force is due to both $\vec{N}$ and $\vec{w}$ according to our FBD

$$\vec{w}_{radial} = mg\cos\theta$$

$$\sum F_r = -N - mg\cos\theta$$

and this must equal F_c. But careful, in these coordinates the direction of the acceleration vector is also in the minus direction!

$$-\frac{mv^2}{R} = -N - mg\cos\theta$$

Now, at the very top of the loop $\theta = 0$ and we have

$$\frac{mv^2}{R} = N + mg$$

Now, we want the normal force to be greater than zero. If $N = 0$ then there would be no contact with the track and we'd be plummeting to our doom! This means we demand the speed to be greater than the situation without the normal force. If we can get to the top with the speed such that $N = 0$ then any speed greater than that will get us over and around.

$$N = 0 \Rightarrow \frac{mv^2}{R} = mg \Rightarrow v^2 = mgR$$

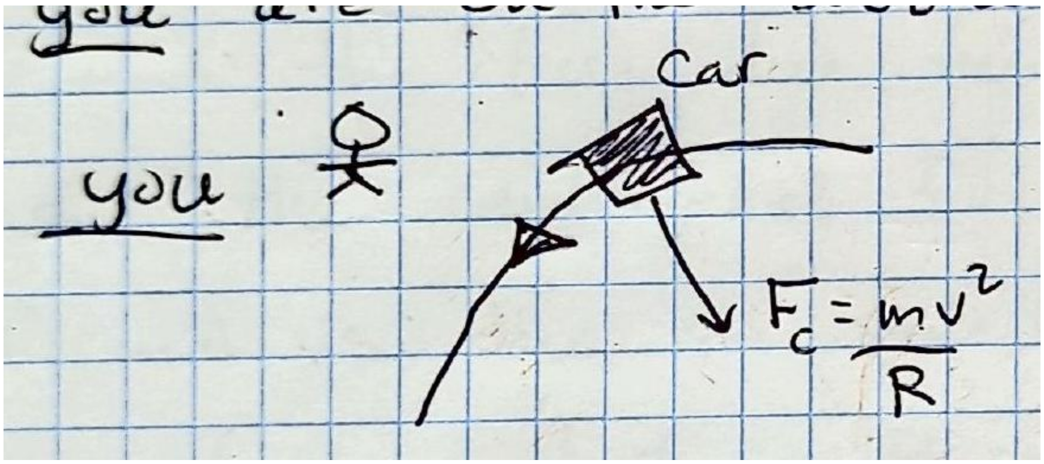
This speed is what we need to maintain contact! Any speed less than this and we are doomed!

Centripetal versus Centrifugal

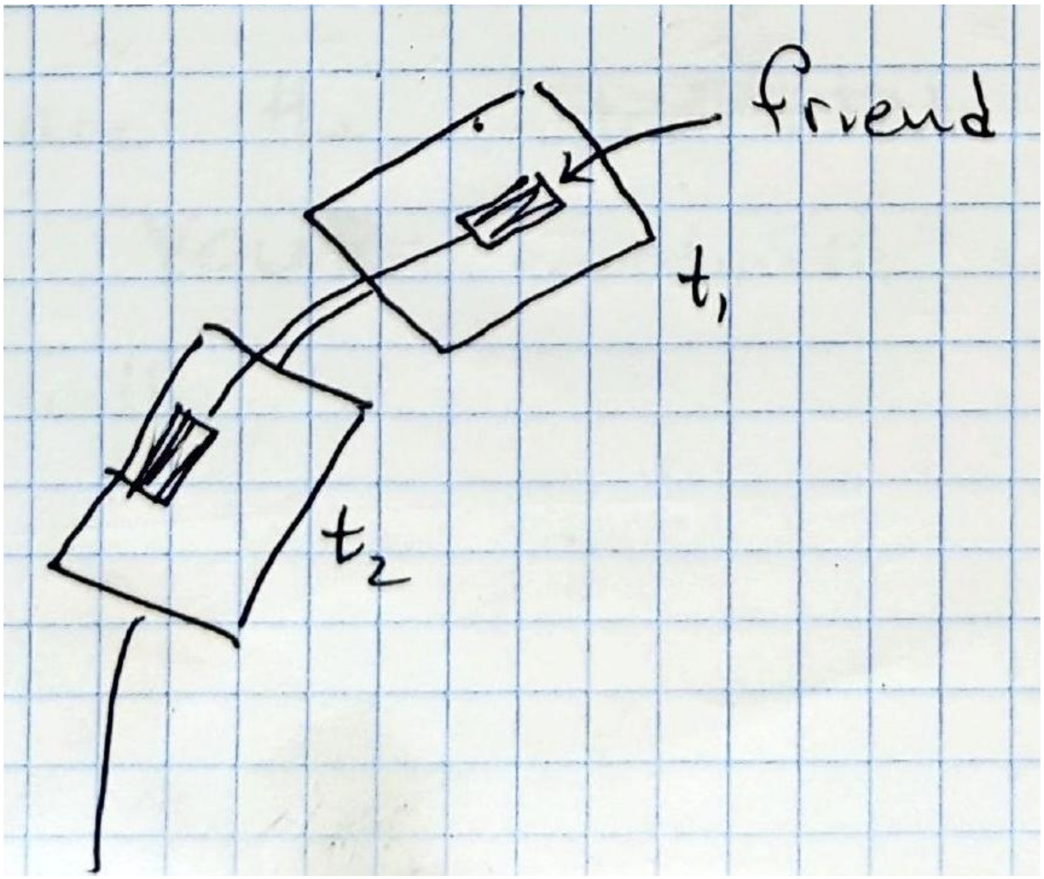
A common difficulty in learning mechanics is understanding the difference between "centripetal" and "centrifugal" forces. It takes a considerable amount of time and effort to retrain yourself to think in terms of centripetal forces.

Ultimately the difference between the two is the question: "whose frame of reference are you considering?"

As an example, consider a car going around a corner and you are on the side walk. You've surely seen such things countless times.



You set up your clocks and rulers next to yourself (coordinates!) on the motionless sidewalk and you observe inside the car your friend get scrunched up against the outside window as the car turns around the bend.



According to your observation you see friction between the car's wheels and the road pull the car around the turn and inside the car you see the friction between your friend's seat and body similarly pull her along. However, he muscles were not prepared to pull her along the turn and so as they give way her body scrunches up next to the door and the door itself pushes her around the bend by its normal force.

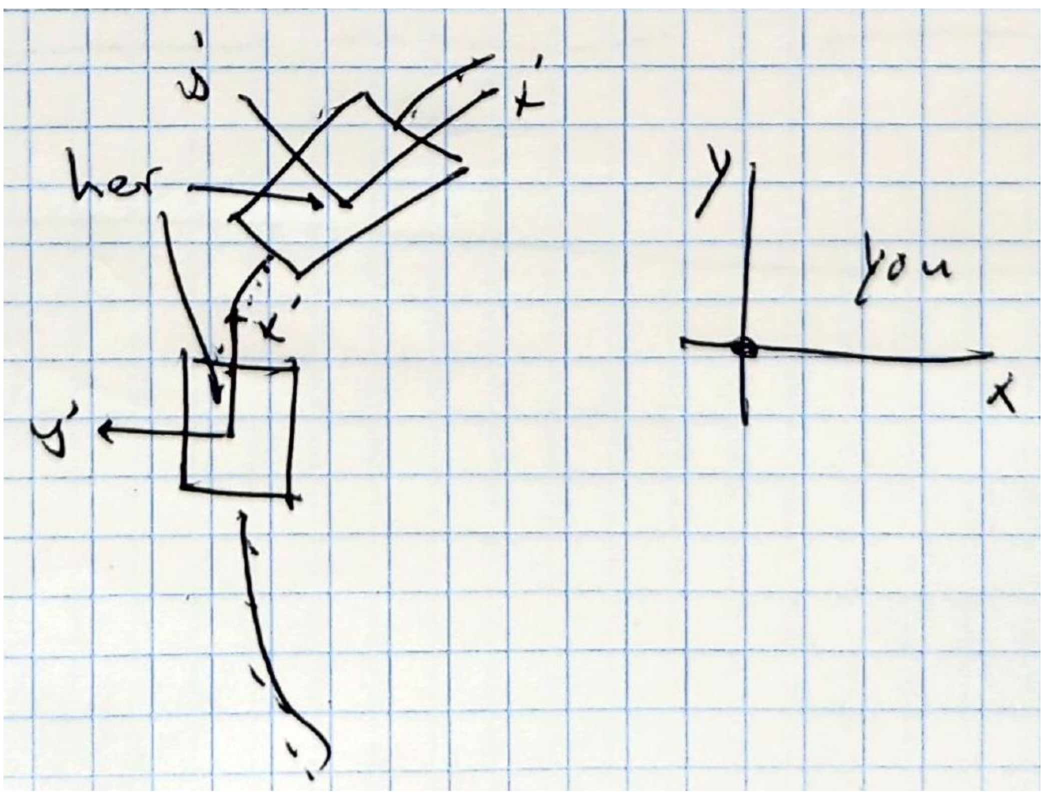
According to you at all times you observe and identify a centripetal force. This force is what's pulling your friend around towards the center of the circular path. Nothing fishy or mysterious about it!

Now, during this time your friend calls you on the phone and explains that according to her and her clocks and rulers (coordinates) she experiences something mysterious pushing her towards the door of the car, i.e. away from the center of the turn! But you explain she's being pulled the other way. So who's more correct here? She literally feels it herself, you only watch it. maybe that means she's correct?

You see a centripetal force pulling your friend around the turn. She's sees an oppositely oriented force pushing her away from the turn. We shall call this force she "feels" a "centrifugal force".

Now surely if all is right in the world of physics these two forces had better be equal and opposite yet we are at an impasse.

The difference ultimately stems from the fundamental differences between your two coordinate systems. According to you, your coordinates are fixed and hers are moving and rotating. According to your friend, her coordinates are fixed and it is YOURS that are moving and rotating. see below



We have a special name for rotating (or accelerating) coordinate systems: they are called **non-inertial**. Rotating coordinates are accelerating. Any time you have coordinates accelerating relative to another set of coordinates one discovers something called "pseudo-forces". They're absolutely not fake...they simply differ from their non-inertial counterparts and they "feel" mysterious. Your friend "feels" a push away from the turn. This is a real push but it is due to her rotating coordinates only. if she could extricate herself and her coordinates then she'd naturally find the normal force doing all the work to bend her around the turn. It is ultimately a complicated application of Newton's third law.

I do not want to give you the impression that there are special coordinates which exist that are purely inertial. In fact, your coordinates on the sidewalk are also secretly non-inertial relative to the sun. Your coordinates on the sidewalk are accelerating relative to the sun. But the sun's coordinates are accelerating relative to the galactic center and on and on!

Our coordinate systems aren't "real"; they're however we choose to arrange our clocks and rulers. What matters is that the physics is the same in all coordinate systems and indeed the magnitude of the centrifugal force is equal to the centripetal force apropos Newton's law.

When solving rotational problems your goal should be to always use Inertial, non-accelerating, coordinates. We wont always be able to, but if you do then everything will be centripetal. Sometimes it helps to choose non-inertial coordinates but one must take extreme care to identify all the pseudo-forces.

Reference Frames

We've seen there's at least a peculiar and fundamental difference between frames rotating or accelerating relative to each other. Here we will discuss frames of references that are inertial apropos each other.

Suppose two people, Alice and Bob, are in motion relative to each other. For definiteness consider Alice, driving down a straight road at constant velocity would have an inertial frame inside the car. Bob, sees her dive by, standing on the sidewalk would also have inertial coordinates.

Bob would measure Alice's velocity to be 30 mph according to the rulers and clocks he has laying about his feet. Alice, however, would find YOU moving 30 mph in the opposite direction since her Frame of Reference (FOR) is attached to her car. After all, she is not mowing according to her own clocks and rulers. According to her she's at $(0, 0)$ and there she stays. If her car had no windows she would not be able to do ANY experiment, in principle, that could inform her that her car is, in fact, moving with some speed.

Now, it is an experimental fact that physics in one inertial frame MUST match the physics in another inertial frame. You, me, Alice, etc must all agree on the results of their experiments. If Alice throws a ball in her car then you and her should ultimately agree about its physical properties! However, if Alice throws the ball while she is accelerating then the results of her experiments will suggest to her a mysterious force that you, outside, would consider to be fictitious. To Bob the centrifugal force is just newtons first law saying that her body wants to travel in a straight line. Alice, inside her car with no windows, we feel a crazy push from the heavens that she wouldn't be able to explain physically whatsoever using her FOR!

Now it turns out that Bob, there, on the sidewalk is also technically non-inertial. Bob's coordinates are rotating with the earth about its axis, they're also rotating around the sun, around the galactic center is so on. These rotations are about a VERY large radius of curvature to be sure so that the pseudo-forces Bob feels would be teeny tiny! But, in principle, with careful enough experiments Bob would find several mysterious forces due to these rotating coordinates indeed by studying

$$F_c = \frac{mv^2}{R}$$

for large R the forces would be small and Bob must be a VERY capable experimentalist to measure them using his sidewalk clocks and rulers!

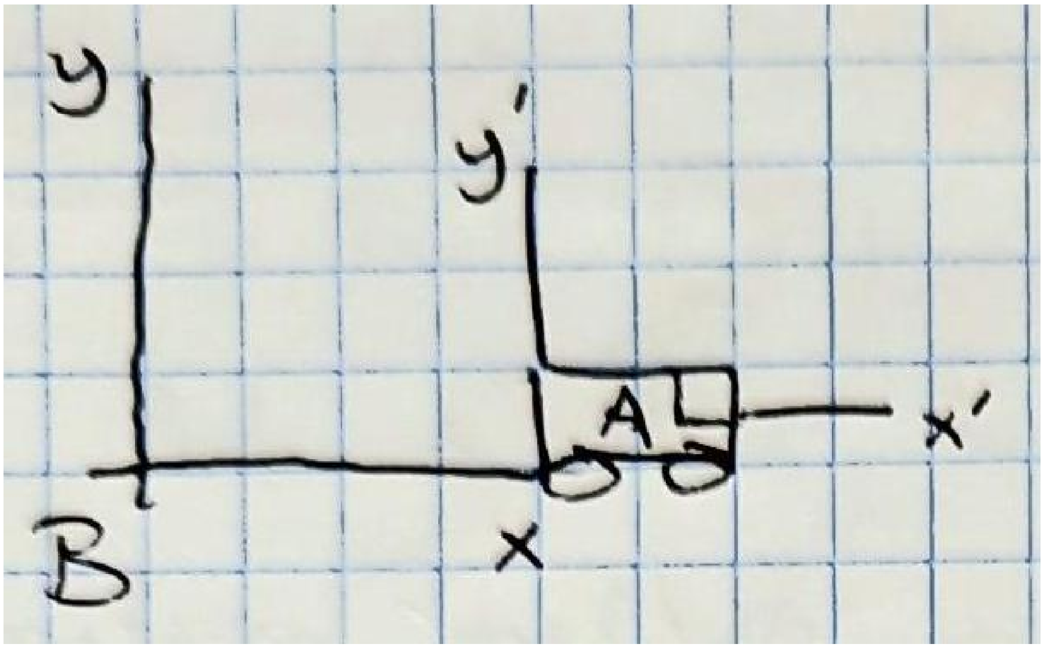
Because of these tiny effects we approximate and say Bob's sidewalk coordinates are inertial inertial enough when viewing Alice driving around. If Alice suddenly turns then the pseudo-forces she feels will certainly be great enough for her to measure!

However, the pseudo-forces do become apparent when dealing with "Earth-scale" distances. For instance, artillery officers and rocket scientists must consider pseudo-forces in their calculations lest they miss their targets by miles and miles.

Let us discuss the connection between two inertial FORs.

Okay, Bob sees Alice drive by at 30 mph from his sidewalk laboratory. He looks inside her car and he can see clearly the rulers and clocks she's set up: her coordinate system. According to Bob the two coordinates look something like the figure below (we are considering that at $t = 0$ the two coordinates are at the exact same place and as time ticks forward Alice's coordinates move to the right along the x-axis at 30mph.

We call Bob's coordinates (x, y, t) and we call Alice's (x', y', t')



Suppose Alice and Bob agree on their y and t coordinates. i.e. $y' = y, t' = t$.

Let's say that Bob measure's Alice at $t = 1$ to be at $x_A = 5$ meters in HIS coordinates.

At the same time Alice will measure Bob's position to be at $x'_B = -5$ meters using HER coordinates.

Suppose now that at $t = 1$ Alice drops a ball from a height $y'_{ball} = y_{ball} = 10$ (she has a very tall car indeed!) Let us record what each person finds:

Alice

According to Alice she takes (at $t = 1$)

$$g = -10m/s^2$$

$$y'_{Ball} = 10$$

$$x'_{Ball} = 0$$

$$\vec{v}'_{iBall} = 0$$

In Alice's coordinates these comprise her initial data and so she could call Galileo and find out what the ball does inside her coordinate system.

When Alice lets go of the ball, she observes it drop vertically by $y'(t) = y'_i - \frac{1}{2}gt^2$ since there is zero initial velocity in her FOR. Plugging in her initial data we find

$$y'(t) = 10 - 5t^2$$

The ball hits her car's floor when $y'(t_f) = 0$ or

$$10 = 5t^2 \Rightarrow t_f = \sqrt{2}$$

This is Alice's physical prediction: it takes $\sqrt{2}$ seconds to be at $y'_f = 0$

Bob

Now, let's see what Bob observes and measures? According to Bob the ball is not only falling but is also moving in his x-direction. This means we have to solve a 2D problem. Let's figure out what Bob knows in his coordinates.

$$g = -10m/s^2$$

$$y_{Ball} = 10$$

$$x_{Ball} = 5$$

$$\vec{v}_{iball} = \begin{bmatrix} 13 \\ 0 \end{bmatrix}$$

Take note that we have converted 30mph into 13m/s.

Now, Bob sees Alice let go of the ball at $t = 1$ and so Bob calls Galileo in order to find the 2D motion of the ball.

$$x(t) = 5 + 13t$$
$$y(t) = 10 - 5t^2$$

According to Bob the ball hits the ground/floor of the car when $y(t_f) = 0 \Rightarrow t_f = \sqrt{2}$

Satisfyingly they both find $t_f = \sqrt{2}$s. At the end of the motion. Alice finds the ball's velocity to be zero but Bob finds the ball's velocity is 13m/s. But Bob and Alice differ by that same velocity so that everything matches up! We would be VERY worried if we found otherwise!

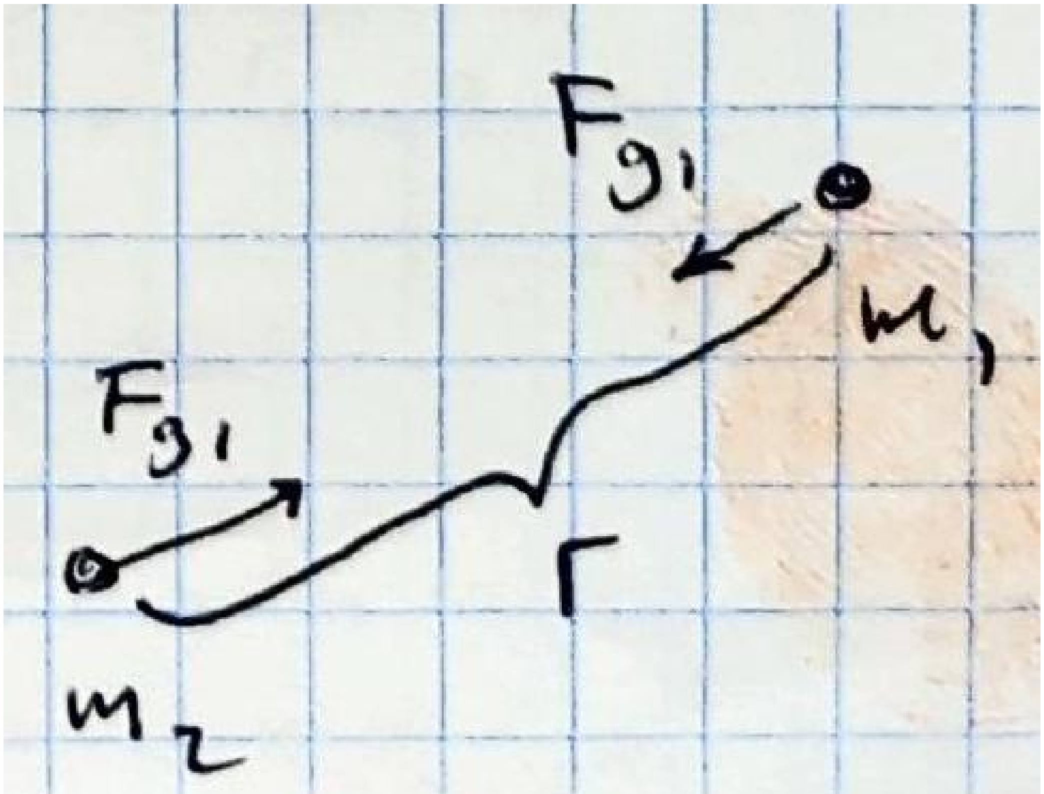
This is a very POWERFUL method in physics called Galilean Relativity. In this paradigm velocities add together so long as the speeds are MUCH smaller than the speed of light (3×10^8 m/s). Funny things happen when velocities get near this value which we won't get into here.

Gravitation

Recall that the force due to gravity between two bodies with mass m_1 and m_2 is given by:

$$F_g = -G\frac{m_1 m_2}{r^2}$$

where r is the straight line distance between the two masses.
As r gets larger and larger force between them gets smaller and smaller and if the distance between the two is tiny then the force can become quite large



In our solar system the force of gravity accounts for the motion of planets about the sun.

If we suppose the earth moves around the sun in a circular orbit then we claim gravity causes the centripetal acceleration of the earth as we have previously seen. Viz,

$$F_C = m_E \frac{v^2}{R} = -\frac{GM_0 m_E}{R^2}$$

where

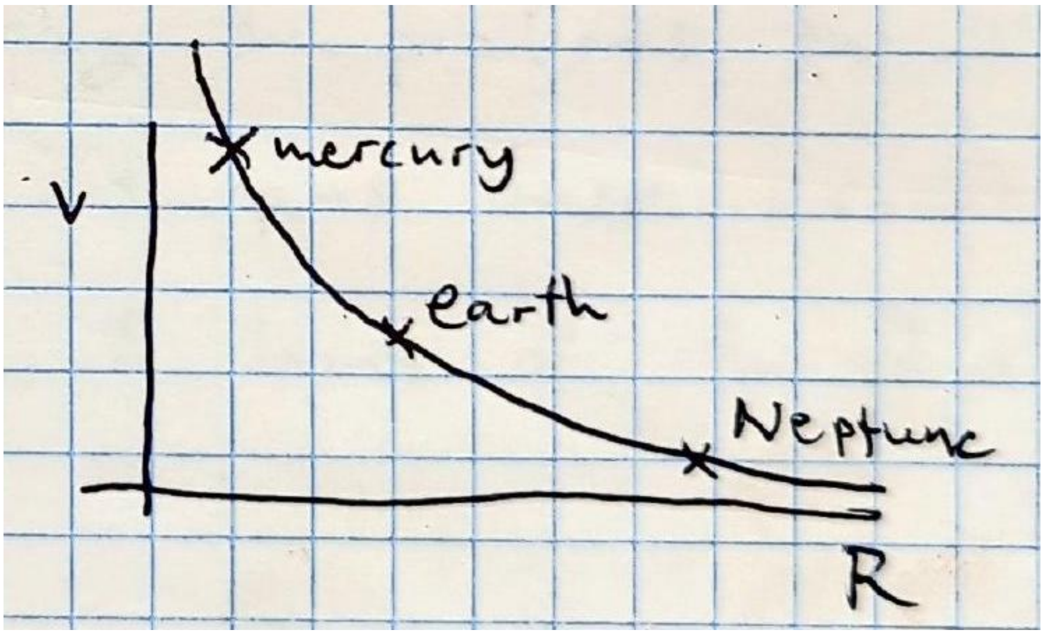
$$G = 6.7 \times 10^{-11} \frac{Nm^2}{kg^2}$$

is called "Newton's Gravitational Constant"

We can solve this equation for v and find that the orbital speed is a function of radius given by

$$v_{\text{orb}} = \sqrt{\frac{GM_0}{R}}$$

It turns out the planets all fall roughly onto this velocity curve. See below



Galaxies behave a little (LOT!) differently. When we ask the speeds of stars from the center of the galaxy we find a much different curve. at medium to large values of R we find the speed is roughly constant and on top of that much too large according to Newton!

It was proposed that a "dark" type of matter in and around the galaxy could account for this and we have been hunting incessantly to no avail despite the fact that we know something is up. It's one of the greatest mysteries in physics

Kepler

Long before Newton reasoned his laws and his $\frac{1}{r^2}$ force philosophers had been perplexed by what caused some of the little dots in the sky to move in weird wiggles that tracked back and forth and looped around whereas others simply didn't. Confusion abound until Copernicus, Tycho Brahe, and Kepler showed up to set things straight.

Copernicus reasoned these moving dots as well as the Earth go around the Sun rather than around the Earth as people had long thought. Upon the invention of the telescope Tycho took to the
task of pain-stakingly measuring and recording the positions of all the heavens to great accuracy for his entire life (interestingly the King of Denmark gave Tycho what amounted to an entire island and 1 trillion dollars worth of gold in today's money! what a grant!) It was these measurements that allowed Kepler to deduce 3 laws of the planets empirically.

1. The Sun must be the focus of an ellipse and the ellipse is the orbit of a planet around the Sun

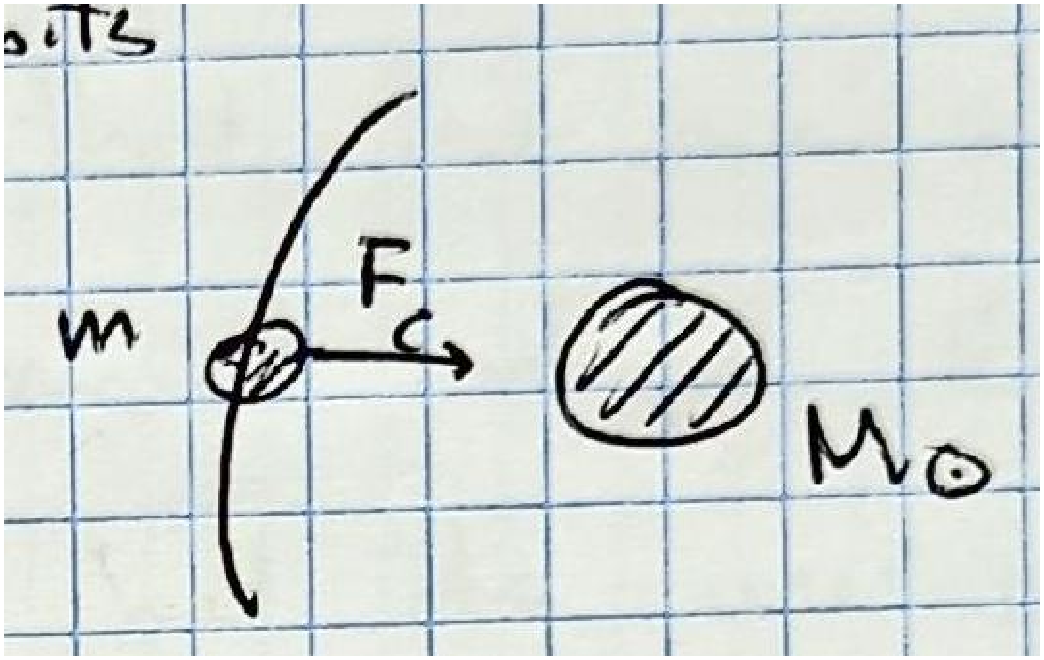
2. When a planet is close to the sun it speeds up such that it traces out equal areas in equal times

3. we can compare 2 planets and find the relation
   $\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$

Where T_i is the period of the orbit and r_i is the semi-major axis of the ellipse.

Newton, with his theories of gravity, forces, and calculus was able to explain all three of these laws succinctly.

We can get a rough idea where (3) comes from by supposing circular orbits (the proof for elliptical orbits is a little trickier).



First, $F_c = F_g$ - the centripetal force is due to gravity, then

$$\frac{mv^2}{R} = \frac{GmM_0}{R^2}$$

Now for a complete orbit

$$v = \frac{2\pi R}{\Delta t} \to \frac{2\pi R}{T}$$

and

$$v^2 = \frac{GM_0}{R}$$

so that

$$\frac{4\pi^2 R^2}{T^2} = \frac{GM}{R} \Rightarrow \frac{T^2}{r^3} = \frac{4\pi^2}{GM_0} = \text{ constant}$$

Therefore, Kepler's 3rd law is shown.

The Superb Theorem

Theorem: Inside a massive spherical shell the gravitational force is 0!

This is assuredly NOT intuitive: suppose the earth was actually a hollow planet with a 12 mile thick crust. This theorem means that if you dug a hole and climbed into the interior you would float around in the same manner as deep space whether you were next to the crust or not. All of that mass would be arranged in just the right manner so as to completely cancel out all the forces at any point on the inside.

This theorem is MUCH more important in the study of electricity and magnetism where it goes by the name "Gauss' Law".

Energy

In these notes we will look at a very strange element of reality called "Energy". Ever since Newton set forth the blooming ideas of calculus, forces, and gravity along with the economic interest in building water wheels that convert a flowing river into the grinding of wheat and grains, researchers and people alike started noticing something important about the specific combination of units $\frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}$

Leibniz was probably the first person to notice there was a sort of "Energy" in a moving system he called "vis viva" which he took to be mv^2. A flowing river has "vis viva" and it appeared to be closely connected to how fast or efficiently a water wheel could spin. Inventors noticed that before the river moved the wheel the river had a particular value of vis viva but the water flowing out always had less vis viva and the difference in these was intimately related to the amount of grain that could be ground.

So inventors, without realizing it, had the seed of an idea that we all take for granted these days: "What's the most vis viva I can extract from the river to produce the most flour". Or, another way "how can I maximize the amount of energy i have available?"

Leibniz was correct about so many things and vastly ahead of his time not just with his vis viva but also with philosophy, mathematics, and computing. It would take over 100 or more years and the development of the steam engine for researchers to catch up with Leibniz' intuition vis viva. his vis viva is now known as "kinetic energy" and equals to $\frac{1}{2}mv^2$. He was spot on aside from the 1/2!

Let us develop the amazing and abstract theory of energy. We begin with the concept and definition of "work".

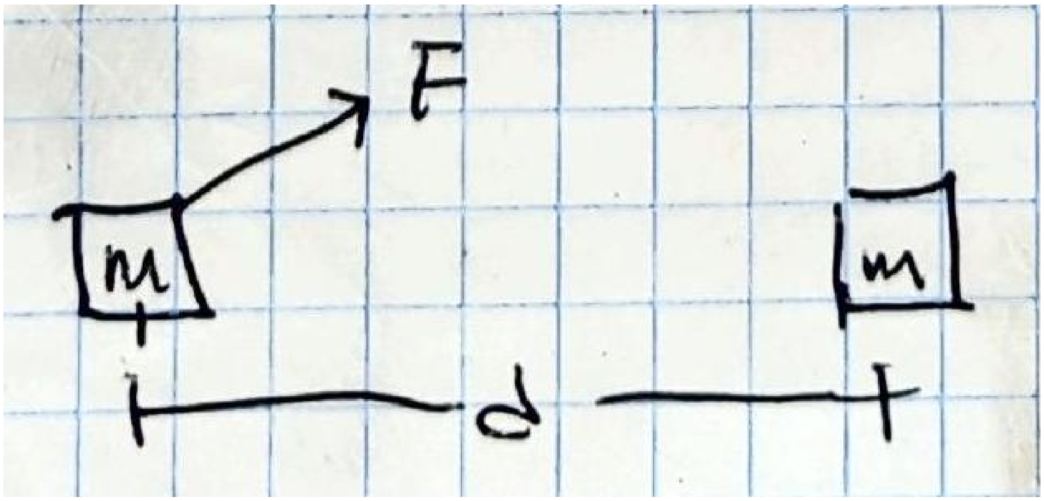
Work

When a force is applied to an object constantly over a distance d we say "work has been done". This is not our colloquial definition of work. Working hard at your desk on a difficult computation is not necessarily "work" in our physical sense (however, inside your body your brain cells are firing more than usual and in different ways thereby causing them to metabolize more and consume extra ATP, etc. I once calculated - based on experimental data - that to strenuously think for 8 hours would require 1 slice of bread. In a sense you do work when thinking - but not the type of work we consider here with our more humble goals!)

The units of work are

$$[W] = N \cdot m = \text{ Joule}$$

In honor of the first person who showed that mechanical energy can be converted into thermal energy. Suppose we have a force applied to an object as such;



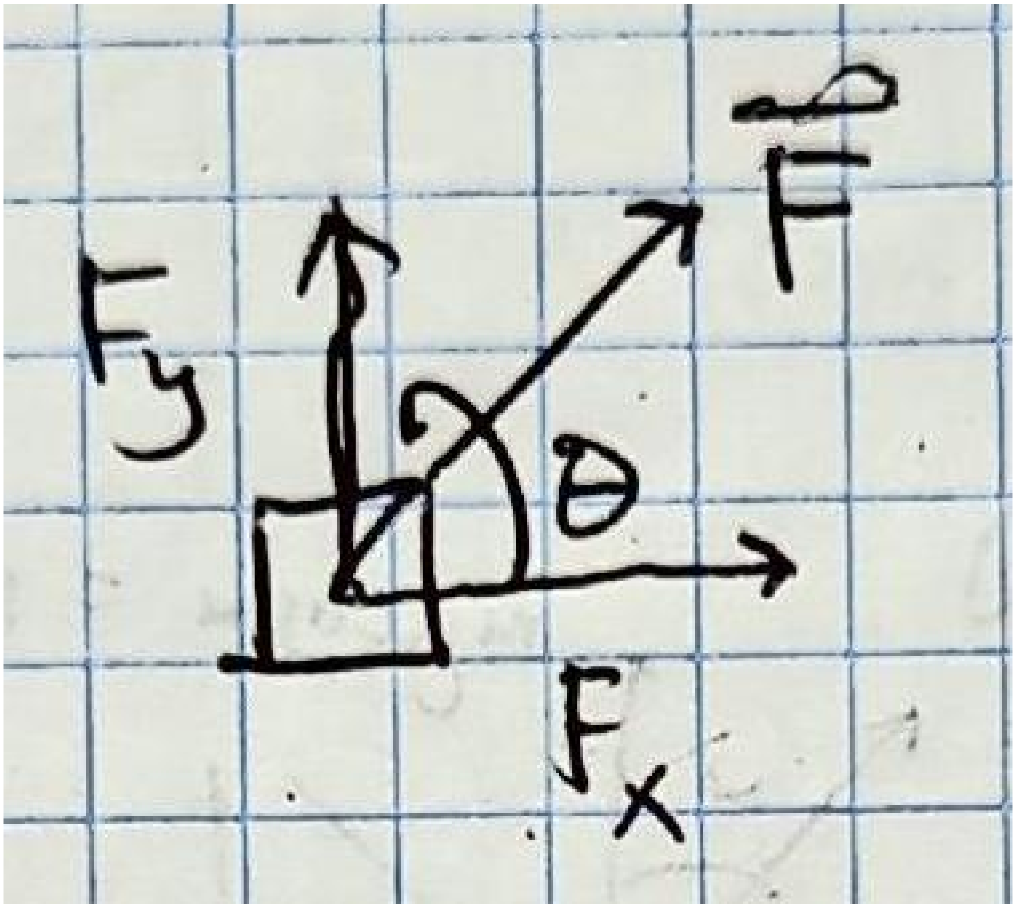
and this force is applied constantly to the mass across a distance d. The force immediately causes the object to accelerate and therefore that acceleration continues until the force in turned off. Hence the mass moves some distance.

Generally the object accelerates in the direction of the force but here we consider a force that doesn't overcome the weight in the vertical direction. So let us suppose the object is constrained to lie on the x-axis alone.

Then we define Work to be the product of the force component in the direction of motion.

$$W = |\vec{F}| d \cos\theta = F d \cos\theta = F_x d$$

see figure below



It is VERY important to remember that any forces or components perpendicular to the direction of motion produce ZERO work! If theta is 90 degrees to the displacement then the cosine makes the work equal 0

Suppose now we lift a weight a distance h from the ground. Well, we can find out how much work we do in this process. The force that we must apply has to be infinitesimally larger than the weight of the mass. So therefore $F = mg$ and the displacement is in the same direction as the force so we find

$$W = mgh$$

This is the amount of work WE have to do AGAINST gravity in order to lift a mass.

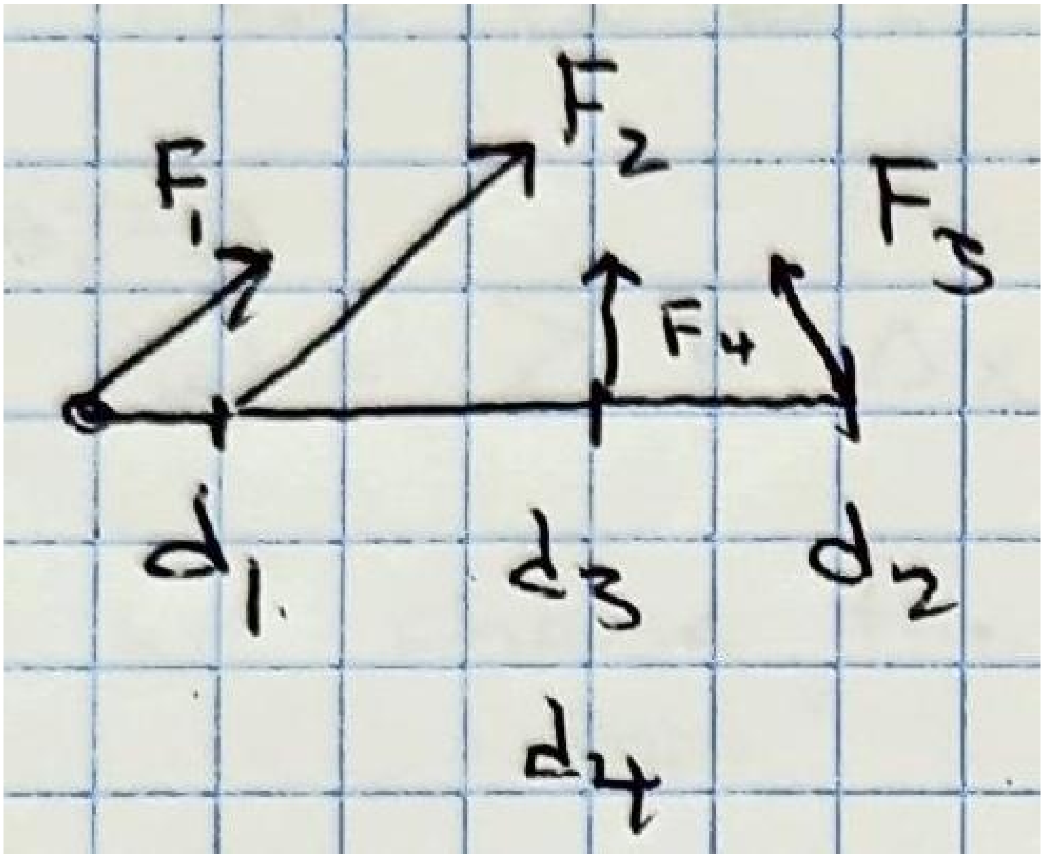
Interestingly, imagine now we hold the mass at this fixed position h. Then our definition says that we are now doing ZERO work! But this is preposterous! Try holding a 30kg mass with your arms outstretched for a few minutes and you'll become absolutely exhausted! You can feel your body suffering and sweating and consuming your energy so how can we possibly say we're doing zero work here?

Well, the resolution is in the fact that we know nothing about how strong you happen to be or even how your body works. The reason we get tired doing such things is fundamentally due to how our muscles and cells and bodies operate. If you put a book on the shelf you do work against gravity but ones its on the shelf it just sits there: no work is being done.

But you object: "there are forces in the wood of the shelf that are holding the book up. Surely they're doing work". and the remedy to this paradox comes down to how the molecules of the wood operate. It's a really good approximation to consider them like perfect springs: when you put the book down, the springs compress a little bit but once they do, then they're done compressing and the net force on the book is zero and these little "electric" springs fundamentally never "get tired".

Subtleties abound! For our purposed we will not consider such Russian Doll questions and simply stick to the definition. If something is not being displaced then the work done is identically zero. And if something is being displace but exactly perpendicular to the force applied then that too is $W = 0$.

Now, suppose an object moves along some path and the amount of force applied to that object varies as in the figure below:

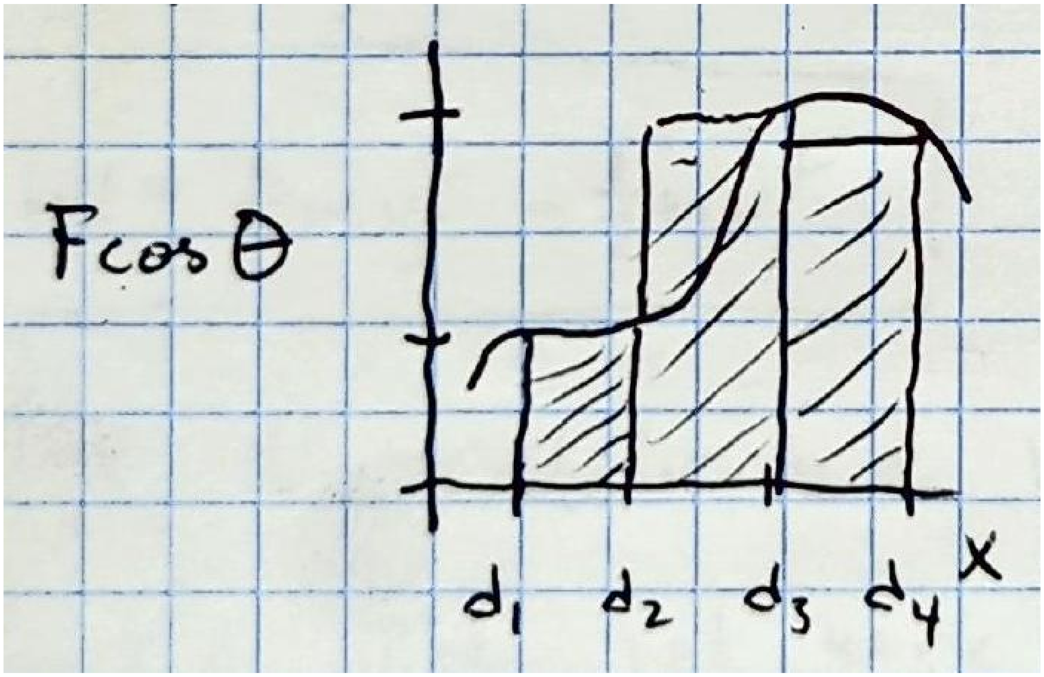


This seems enormously complex but thankfully we are able to break this motion up into individual pieces and add them all up.

First, along d_1 we have $W_1 = F_1 d_1 \cos\theta_1$. But then we can do the same for d_2 and d_3 and on and on. Therefore we have

$$W_{total} = \sum W_i = (F_1 \cos\theta_1)\, d_1 + (F_2 \cos\theta_2)\, d_2 + \ldots$$

We can plot this as follows (take $F\cos(\theta)$ as our vertical axis and distance as the x-axis



In our figure we allow that the force is varying continuously.

The product $\Delta F \Delta x$ is the work for that motion along that increment. It is the area of the rectangle given by Δx and ΔF.

Now, if we take very small Δx's then the rectangles will get very narrow and will approximate the area under the curve if we add them all up.

The area under this curve is the total work as $\Delta x \to 0$. Methods in calculus make this idea precise. For our purposes we shall not be considering continuously varying forces.

Now, there's an incredible fact that we can pluck from these ideas and what we've previously learned.

Recall, we found

$$W = \sum_i F_i \Delta x_i \cos\theta_i = \sum \left(F_i \cos\theta_i\right) \Delta x_i$$

For simplicity let us consider equal increments and equal forces which happen to all be in the direction of motion, viz, $\theta_i = 0$. It will be your task to prove the following in the general case.

Then, plugging in our assumptions, we have

$$W = \sum F_i \Delta x_i$$

but we choose the $\Delta x_i = \Delta x$ to all be the same. therefore

$$W = \Delta x \sum F_i$$

But $\sum F_i = F_{net}$. Therefore

$$W = F_{net} \Delta x$$

but Newton says $F_{net} = ma$, then

$$W = ma\Delta x$$

but recall from Galileo

$$v_f^2 = v_i^2 + 2a\Delta x$$

then

$$a = \frac{v_f^2 - v_i^2}{2\Delta x}$$

Now we can plug this into our expression for the total work

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

And lo, we arrive at one of the single most important ideas in all of physics: the celebrated work-energy theorem.

This expression says: The net work ON a system is given by the change in this new quantity $\frac{1}{2}mv^2$ that looks a LOT like Leibniz' vis viva!

We call $\frac{1}{2}mv^2$ the Kinetic Energy (KE) of the mass.

It is the "energy of motion" (though we have no idea yet what energy even is, this "thing" we've found seems important).

If we know the work done on an object the we can use this information to find its speed! Another way of looking at it is to say "doing work causes things to move". Well, duh.

Now, here, you object that we have made some dubious assumptions indeed!

However, the best I can tell you right now is that careful and general analysis with calculus shows this holds generally for any kind of force and motion.

Now, let's recall our lifting of a mass (I do hope you're not still holding it there!). The work YOU must do to lift a mass to a height h is

$$W = mgh$$

Here I highlight a BIG WARNING: when discussing work you ALWAYS have to consider "who" is applying the force. This is a Newton's third law type consideration. In the case of lifting the mass i could ask

"What is the work done by gravity when you lift the mass from the ground to your head?"

This is MUCH different than asking

"What is the work YOU do when lifting the mass from the ground to you head?"

What's the difference you say? Well, YOU have to apply a force AGAINST gravity in order to overcome gravity. The force YOU apply is in the same direction as the motion of the mass.

Gravity meanwhile applies a force in the OPPOSITE direction to the motion of the mass. Therefore (due to the $\cos(\theta = 180) = -1$ we find that gravity does NEGATIVE work compared to yourself

$$W_{you} = -W_{gravity}$$

So it ALWAYS matter who you are thinking about as applying the force.

Now, we usually write this as

$$PE = U = mgh$$

Generally, the potential energy (PE) of an object is due to its relative position with respect to another body which provides a force.

For gravitational PE it is due to the force of gravity from the earth/mass system. The distance " h " is the separation in distance between the surface and the mass. The gravitational PE we have defined above is ONLY applicable near the surface of a planet. It is an approximation that we use in our daily lives but if we were sending rockets out to space then we absolutely must use the completely accurate form

$$PE_m = W_{you} = \frac{GMm}{r}$$

This is the GPE of the small mass relative to the big mass, e.g. the mass of the Earth.

It turns out that we can choose the surface of the earth to be at $h = 0$. But this seems to imply that at the surface of the earth the PE is also zero. But according to the more accurate formula above the PE is assuredly not zero at this point. It would only be zero if we took $r \to \infty$. So what gives?

It turns out being a matter of coordinate choice along with a VERY important fact in physics: it doesnt matter what the potential energy is, it ONLY matters how it changes. i.e.

$$PE_f - PE_i = \Delta PE$$

It is this change in PE that determines much of a systems behavior.

the specific value you assign to a particular place is a relative choice. For example, a book on a shelf on the ground has a PE relative to the surface of Earth (ground).

However, if the same shelf is in a skyscraper then we may not be too concerned with its motion relative to the earth's surface (unless we push it out the window. We may wish to choose our 41$^{\text{st}}$ floor floor to be the zero-point: That is to say, we measure all the motion relative to the floor way up high in the sky. In that case, if the book falls, the claim is that it will fal in precisely the same manner it fell when the shelf was on the surface of the floor. The heights relative to the ground are equal in both situations. i.e. $\Delta PE = mg\Delta h$ where Δh is the height of the shelf.

Notice that potential energy can differ by a constant amount. When we compute $\Delta PE = PE_f - PE_i$ then if I add a constant amount C to the potential energy formula I get $\Delta PE = (PE_f + C) - (PE_i + C) = PE_f - PE_i$ - nothing changes. weird

Now, suppose we lift a 1kg ball to a height of 10m with respect to our chosen zero point (the ground). Then we can ask: "what is the ball's speed when it hits the ground".

Well the change in PE is

$$\Delta PE = PE_f - PE_i = 0 + 10mg = 10mg$$

Generally $\Delta PE = -W$

Here, we must be double-plus careful. This is gravity doing the work, hence the plus sign!

Now, the work-energy theorem says that

$$\Delta PE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

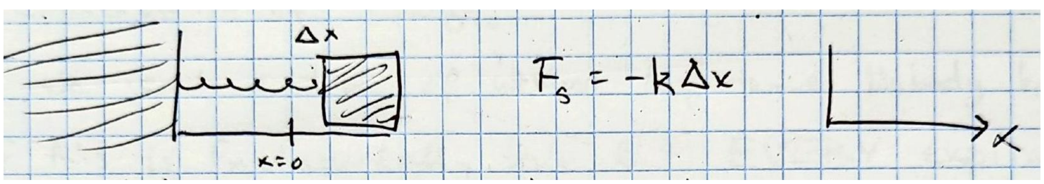
but $v_i = 0$ since we dropped it from rest. So we have

$$100 \text{ N} \cdot m = \frac{1}{2}mv_f^2 \Rightarrow v_f^2 = 20\frac{m^2}{s^2} \Rightarrow v_f = 2\sqrt{5} \text{ m/s}$$

No Newton, no Galileo in sight. Just a quick computation involving kinetic and potential energy. Whatever energy is, it certainly makes things easier to compute!

Now, it has been found that various systems have various equations that give their PE just like we had many different equations modeling forces of springs, friction, gravity and more.

Suppose we have a mass attached to a spring in we stretch it a distance $\Delta x \neq 0$



Recall that a spring opposes a change in x. If we stretch it by Δx we can find the work done by the spring (or ourselves) during this stretch

Now, $F_s = -k\Delta x$, then

$$W = F_s \Delta x = -\frac{1}{2}k(\Delta x)^2 = -\Delta PE_s$$

The $\frac{1}{2}$ occurs because we technically stretch a finite amount and therefore should consider the average force from 0 to Δx. Calculus takes care of this subtlety for us.

So we have found that the potential energy function for a spring is given by

$$PE = \frac{1}{2}k(\Delta x)^2$$

This is the work YOU do on the spring when you stretch it by Δx. All that matters is the stretch and not the path it takes.

This PE_s is one of the premier formulas in ALL of physics. It shows up in nearly EVERY system imaginable for quite profound reasons we wont get into. We will dig only into the surface of its utility and beauty!

It is amusing to note the functional similarities of KE and PE_s

$$\begin{cases} PE_s = \frac{1}{2}mx^2 \\ KE = \frac{1}{2}mv^2 \end{cases}$$

Perhaps this is something deep!

Conservation of Energy

We have now reached a topic of utmost and supreme importance. Nobody knows if what we shall soon learn is fundamentally true but EVERY experiment EVER done (when analyzed carefully and reproduced) conserves its total energy. This means the sum total of all the energy of a system never changes. We can take one step further and claim that the total energy content of the universe itself is constant!

We have many mathematical and theoretical reasons for why this "should" be true but ultimately it is up to nature herself.

Every time an experiment has seemed to violate this conservation law a new form of energy has been found lurking in the shadows. It is so solid and consistent that we call it a LAW. To show a violation would require a FANTASTIC amount of evidence before a single eyebrow would be raised!

The law goes something like this:

If we perform an experiment and a bunch of masses move around or change or whatever, then it is the case that if one adds up ALL the energies before the experiment, and all the energies afterwards then we will find the two values are the same.

For example, imagine I walk into a room with a box of calm hornets. I have previously found their total energy to be some small value. Some are buzzing about here and there but they're relatively calm. Now, I place the box on the table and leave the room to eat some lunch. Upon my return I open the box and to my surprise all the hornets bolt out in all directions with their stingers ready to pummel my face into oblivion.

I therefore conclude, immediately, that you dear reader, have shaken by box. You have taken some of your own energy and you have dumped that into my box thereby increasing its energy.

But you exclaim, as the good student you are, that you had taken careful note of the total energy inside the room before you left. You had even texted me the evidence of your computations. You implore me to measure the total energy inside the room. When I do I discover the room has more energy in it than before I left the room!

Dear reader, what does this tell me?

It means the only possibility is if someone snuck in while I was getting lunch, deposited some of their energy (which came from the outside!) and then snuck back out. Therefore the conservation of energy for the room alone is violated unless I track down that rogue who shook my box

Conservation of energy is often a detective game: steadfastly searching for culprits. This little story is EXACTLY how conservation of energy works. Every time we have deeply searched we have found the culprit!

Now, consider the work energy theorem

$$W_{net} = \Delta KE$$

But we know that this net work was the change in potential energy. It is the work YOU did in lifting the book or stretching the spring. So

$$\Delta PE_{you} = \Delta KE_{system}$$

But we wish to write everything in terms of only the system - independent of us. Well Newton told us his 3rd Law

$$F_{you} = -F_{system}$$

Then

$$W_{\text{you}} = -W_{\text{sys}} = -\Delta PE_{\text{sys}}$$

$$-\Delta PE_{\text{sys}} = \Delta KE_{system} \Rightarrow \Delta KE + \Delta PE = 0$$

Let us call $KE + PE$ the "mechanical energy". We will, for brevity, just call it "the energy E". Then

$$\Delta KE + \Delta PE = \Delta E = 0$$

This means that the energies before are equal the energies after.
No matter what happens, so long as nobody sneaks some energy into your system.

It also means that PE and KE can morph into each other during a process. In fact, it's even deeper than that! Energy can transform in to all sorts of different types. If a process starts with 20 Joules of energy being strictly kinetic then during some complicated process it's possible that all of that energy gets transformed into other types: chemical energy, thermal energy, radiation, nuclear, gravitational, and on and on. But whatever happens, when you add it all back up you should always get back your 20J of energy. Otherwise you've either lost some or someone has given you extra!

We have open a box and an entire universe stands before us! This is one of the most useful ideas in all of science!

Our problem solving strategy becomes some thing like this:

Set up your experiment and write down all your known quantities. e.g. things like initial masses, speeds, distances, etc.

Then start your experiment and measure your results. You can use Conservation of Energy (COE) to predict final masses, speeds, etc.

A curious feature of COE is the neglect of any notion of time (outside of signifying initial and final parts of your problem - though you need not any explicit value of time). The reason time doesn't show up in these computations is enormously deep.

It was discovered in the early 1900's by the great mathematician Emmy Noether that COE is the consequence of a fundamental symmetry of time.

Whether I use my own clock or you use your own should produce the same physics in a reasonable universe. Furthermore, i could set up my experiment today or 215 million years in the future and i expect to get the same reasults (unless somebody fiddles with the total energy of the universe!).

This realization Had ASTOUNDING consequences for the direction of physical theories. It is called Noether's Theorem who says: any time you have a symmetry there will HAVE to be some conserved quantity.

time $\iff$ energy
space $\iff$ momentum
rotation $\iff$ angular momentum
quantum phase $\iff$ charge

and many more!

Conservative and Non-Conservative Forces

It happens that there exist two general types of Forces.

1. **Conservative**: The net work does not depend on the path the system takes. ONLY initial and final points matter (e.g. gravity, springs, etc)

2. **Non-Conservative**: The work done fundamentally depends on the path taken. The main example is friction

Any time we have friction some energy will be dissipated. This dissipation is what we call thermal: Friction causes atoms to jiggle and that juggling jiggles other atoms and those jiggle others still and on and on. As you can imagine the energy begins "spreading out". Initially it was all in one spot and then over the course of the process radiated away. Typically, there's so many darn jiggling things that we loose track of our energy and then say that "we lost energy due to heat". If you could somehow keep track of every single thing in your system then you could claim you didnt lose any energy

Well, it's really really hard to keep track of EVERY 10^{23} molecule in a room so we say "energy is not conserved" during friction.

The tendency for energy to spread out is called "entropy" and a law of the universe says "entropy always increases in a closed system". One is always free too gather up all the lost energy and put it back but to do so would require energy itself!

Now, I think you can convince yourself that the amount of heat due to friction manifestly depends on how long the path is. Proof? rub your hands together to make them warm. This path starts at A goes to B and then goes back to A again and repeats over and over. Rub your hands together starting at A and then finally ending at A again and depending how many cycles you repeat you will generate more and more and more heat. If it didn't depend on the path then since you start and end at A you would've done ZERO work. But we have heat! so work must have been done!

Non conservative forces (NCFs) "eat" up energy. They reduce speed, etc. A car won't roll forever because friction eats up the kinetic energy of the massive car. the KE turns into heat.

If you bounce a rubber ball it will not reach its previous height due to a variety of NCFs (friction, drag, etc).

A conservative ball bouncing experiment would bounce up and down forever bouncing converting KE into PE and back into KE over and over forever always reaching the exact same height.

This makes it feel like there are probably exactly zero conservative forces in the universe. However, nature seems to make use of them: gravity, electricity and magnetism, strong nuclear, and weak nuclear forces are considered conservative and there's zero indications they aren't.

Also, while not true whatsoever, for our purposes we will consider springs to be conservative. Springs in the real world generate heat as they stretch and compress but we will consider perfect springs only.

Now, you can compute the energy consumed in such a NC scenario by computing the change in total energy before and after a bounce.

Suppose I drop a ball from a 10 meter height and I find that it reaches a maximum height of 8 m after just one full bounce. I claim this is all I need to know to figure out how much heat was created in this process! Now, you'll notice I haven't at all defined what the heck heat even is and I claim we don't even have to know anything about heat.

Schematically I have the following process. I consider my initial situation to be the instant before I drop the ball and my final situation to be when the ball is exactly at its maximum height after the bounce.

Then,

Initially

$$E_i = KE_i + PE_i = \frac{1}{2}mv^2 + mgh_i$$

$$= 0 + 10mg$$

Final

$$E_f = KE_f + PE_f = \frac{1}{2}mv^2 + mgh_f$$

$$= 0 + 8mg$$

Now, clearly $10mg \neq 8mg$. COE is violated! so that $E_i \neq E_f$

So, let's FORCE COE by pretending we kept track of all the heat that was generated. I will call it Q.

then COE says

$$E_i = E_f + Q$$

This must be true because we're claiming that Q is everything we lost. so then $Q = -\Delta E$!

We find that We have lost $2mg$ due to heat. And therefore each bounce consumes roughly $2mg$ units of energy (Joules) and we might guess the ball will bounce just 5 times. (Caveat: the heat lost is actually not equal each bounce!)

Another way to view this is to say that the heat Q is the work done by NC forces in our problem. I prefer to think of heat generated and you may prefer to think of work done by NC forces. At the end of the day they're the same thing. NC forces do work to produce heat. In this second way of viewing you would write

$$E_i + W_{nc} = E_f$$

which is the same thing as imagining heat generated vis a vis

$$E_i = E_f + Q$$

For a non-conservative process we can define an efficiency as

$$\frac{\text{energy out}}{\text{energy in}} = \frac{W_{out}}{E_{in}} = Efficiency$$

If we burn coal we will assuredly not get all of its energy out.

Our turbines used to extract the thermal energy themselves have friction and on top of that we have this fundamental spreading of energy that we can never get rid of. Because of this we cannot make perpetual motion machines.

Current solar cell technologies convert about 10-20% of sunlight into electricity. A big problem is how to collect and use ALL wavelengths of light rather than just a narrow band - e.g. just the colors we see.

Electric motors however are extremely efficient, they are around $95-98\%$. This is partly because light itself is massless and electricity and magnetism is fundamentally conservative!

Power

Often in our lives we are told to "work faster!". Perhaps it actually is a useful thing to consider. We want to understand how much work can be done per unit time during a process. For example, an electricity plant might want to produce 100 million Joules of energy per second. Let's develop this idea.

We shall define power as the amount of work (or energy) produced per unit time. This will have units of Joules per second which will will hitherto call "Watts".

$$[P] = J/s = N \cdot \frac{m}{s} = [Fv] = watt(W)$$

This is the rate at which work is done. Sadly, we use both W and W for both watt and work so dear reader please take care not to get tangled up here. For your ease I shall write Watt using script font $\mathscr{W}$ but this is not standard.

$$P = W/t \quad 1\mathscr{W} = 1 \text{ J/s}$$

To find power one can use the following equations

$$P = Fv\cos\theta$$

This is the same as $\frac{F\Delta x\cos\theta}{\Delta t}$

A force applied at a certain speed will produce power. More speed $\Rightarrow$ more power. Easy as that

Another equation that's often used for Power is

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2}mv^2 + PE}{\Delta t}$$

and so on.

Power is everywhere. It's how we describe "Power Plants".

A Nuclear Power Plant produces gigawatts of power.

$$1G\mathscr{W} = 10^9\mathscr{W}$$

A billion Joules every second! It uses nuclear energy to heat water into steam which turns electric generators.

Coal Plats are megawatt scale - 1 million joules per second. A nuclear plant can be roughly 1000 times more productive than a coal plant!

You, yourself, are producing $\sim 100\mathscr{W}$ of Power. Most of it is produced in your brain and radiates as heat or used to move muscles and breath and digest and cellular processes all of which is aimed at trying to prevent your energy from spreading out.

Now if $P \equiv \frac{\text{energy}}{\text{time}}$ than Energy $= Pt$

Your house probably has an electric meter that counts $k\mathscr{W} \cdot hr$'s. This is the energy youre drawing from a nearby power plant. Lots of fun things car be computed such as how much it costs to do your homework at home?

You consume $100\mathscr{W}$ at rest and a little more when crunching numbers, If it takes you an hour to do your homework then your energy is $0.1k\mathscr{W} \cdot hr$. Find your local rates and you can see how much you spend on equivalently on homework. In Leander, TX we pay 11 cents per $k\mathscr{W} \cdot hr$

Each year humanity uses more and more energy. By 2040 we are estimated to consume $740 \times 10^{18} J$ of energy each year. This estimation was made prior to the emergence of large language models and artificial intelligence - both of which require a TON of energy!

Our energy use is exponentially increasing with precious few accessible sources in sight.

In the year 2000 we consumed 394 ExaJoules of energy. In 2020 we used 556 ExaJoules! In the next 20 years we need to come up with about 200 ExaJoules of energy - almost as much as we used in 2000!! Any ideas?

$$394 \to 556 \to 740 \to \quad ?$$

Momentum

Recall our definition of momentum

$$\vec{p} = m\vec{v} \quad [p] = \frac{kg \cdot m}{s}$$

We call this "linear momentum" to distinguish it from its rotational cousin "angular momentum".

Since $\vec{v}$ is a vector momentum is as well.

We previously fibbed about Newton's 2^{nd} law. The more accurate statement is that A net force induces a change in momentum and vice versa. Astronauts and rocket scientists are keenly aware of the difference. In deep space if you're out of propellant and need to get back to the mothership then you can change your mass in order to impart a force towards the ship.

$$\vec{F}_{net} = \frac{\Delta \vec{p}}{\Delta t} = m\frac{\Delta \vec{v}}{\Delta t} + \vec{v}\frac{\Delta m}{\Delta t}$$

If your system is not changing its mass then $\frac{\Delta m}{\Delta t} = 0$

$$\vec{F} = m\frac{\Delta \vec{v}}{\Delta t} = m\vec{a}$$

the same as we studied previously.

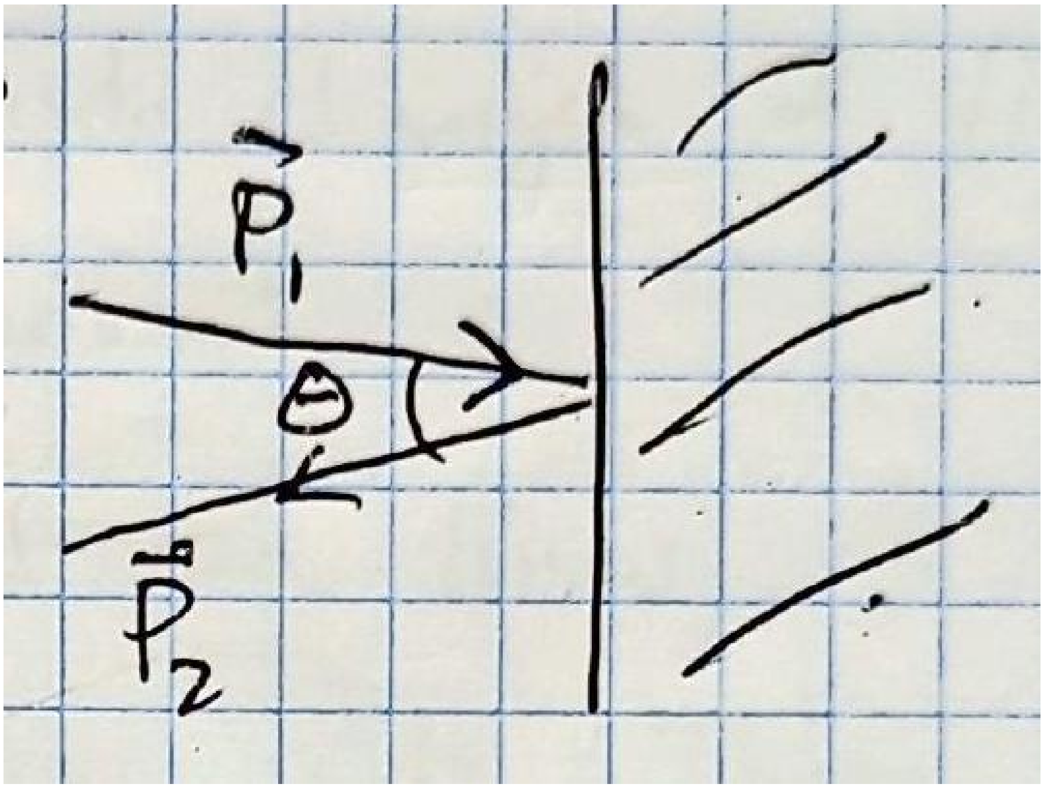
Suppose we throw a ball against a wall. The ball initially has momentum

$$\vec{p_1} = m\vec{v}_1$$

After it bounces off the wall (if no other forces are present) then, for $\theta = 0$

$$\vec{p_2} = -\vec{p_1} \quad |p_1| = |p_2| = p$$

$$\therefore \Delta \vec{p} = \vec{p_2} - (-\vec{p_1}) = 2\vec{p}$$



The wall changes the momentum of the ball and therefore, via Newton, produces a Force on the ball.

We can re-write Newton's 2^{nd} Law as

$$\Delta\vec{p} = \vec{F}\Delta t$$

$\vec{F}\Delta t$ is called the "impulse" . The "impulse" of our ball on the wall is given by $\Delta\vec{p} = \vec{p}_1 + \vec{p}_2 = 2\vec{p}$

If you have a very large force $\vec{F}$ causing a $\Delta\vec{p}$ to go from some value $\vec{p}$ to $\vec{0}$ (say a car crash) then by increasing Δt to be as long as possible you can reduce the Force the occupant experiences. Airbags allow for Δp to change over a very long time thereby saving countless lives.

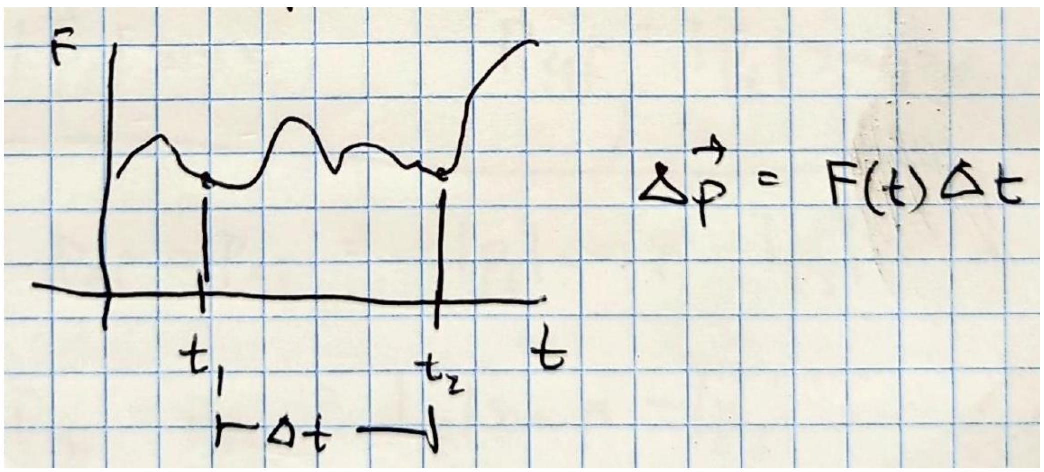
Generally, a longer collision time implies a smaller Force for the same change in momentum.

We can often find momenta changes or forces by measuring Δt in an experiment.

Suppose we know the force $\vec{F}$ at all times and we measure a process to occur over some time range Δt. Then, we can find the momentum change.

$$\vec{F}\Delta t = \Delta\vec{p}$$

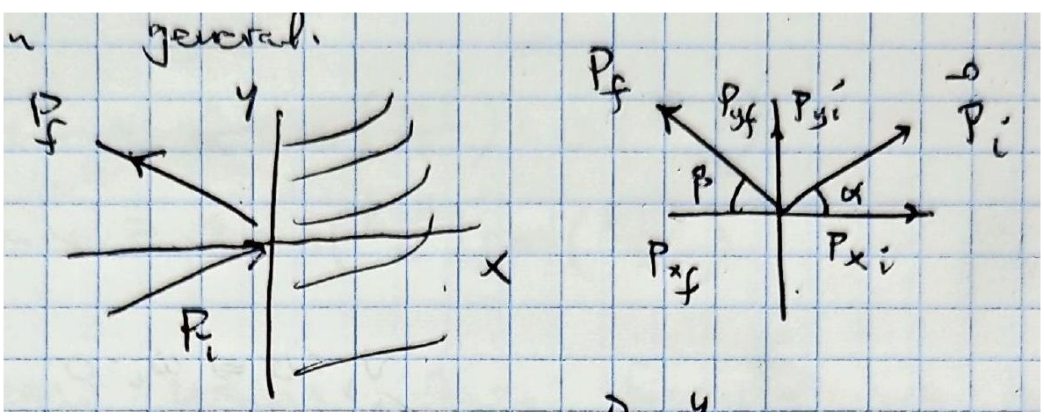
We might have a plot for how the force varies over our time interval as below:



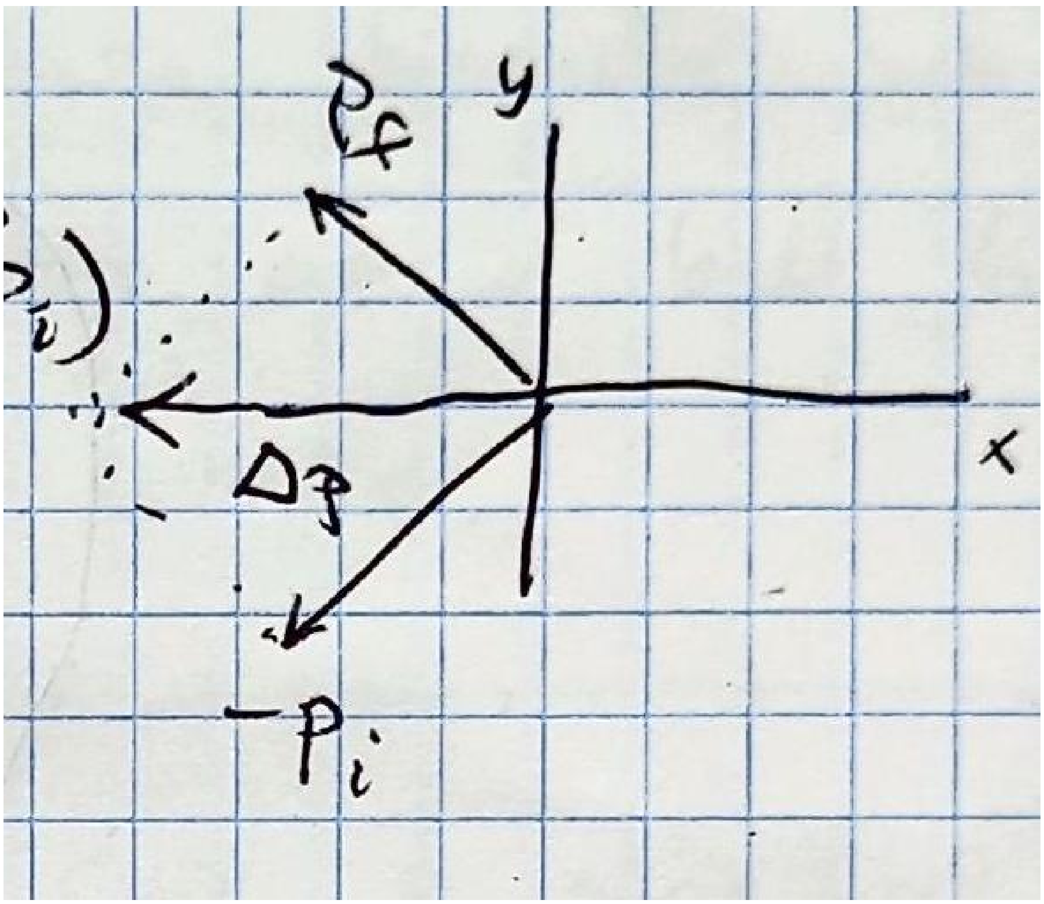
Then the impulse for this process is the area under the curve between Δt.

Often we solve problems by separating a process into before an event (initial) and after (final) an event. Then $\Delta p = p_f - p_i$

Since $\vec{p}$ is a vector we will have components in general.



$$\Delta \vec{p} = \vec{p}_f - \vec{p}_i$$
$$= \left[\begin{array}{c} \Delta p_x \\ \Delta p_y \end{array} \right]$$



but,

$$\begin{array}{ll} p_{xi} = |\vec{p}_i| \cos\alpha & p_{xf} = -|\vec{p}_f| \cos\beta \\ p_{yi} = |\vec{p}_i| \sin\alpha & p_{yf} = |\vec{p}_f| \sin\beta \end{array}$$

so that

$$\Delta p_x = p_{xf} - p_{xi} = -|p_f| \cos\beta - |p_i| \cos\alpha$$
$$\Delta p_y = p_{yf} - p_{yi} = |\, p_f \,| \sin\beta - |p_i| \sin\alpha$$

Now, suppose $|p_f| = |p_i| = p$
Then these equations will simplify to

$$\Delta p_x = -p(\cos\beta + \cos\alpha)$$
$$\Delta p_y = p(\sin\beta - \sin\alpha)$$

We might want to use the trig identities below for further manipulations

1. $\cos\mu + \cos\alpha = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta-\alpha}{2}\right)$
2. $\sin\beta - \sin\alpha = 2\sin\left(\frac{\beta-\alpha}{2}\right)\cos\left(\frac{\beta+\alpha}{2}\right)$

and so on. Using these formula we can solve the general problem of a ball bouncing of a wall at any angle. It all boils down to choosing reasonable coordinates and then decomposing all momenta.

Recall that we previously discussed COE and Nöether's theorem. We claimed that being at different places should not result in an experiment producing different results. e.g. bouncing a ball here is the same a bouncing it in Pakistan, etc.

This symmetry is responsible for another conservation law in physics: conservation of momentum (COM). This COM when combined with COE is incredibly powerful for solving quite complicated problems.

Suppose we have two objects which collide and bounce off on different trajectories. Our claim is that if we add up ALL the momentum BEFORE the collision and ALL the momenta after the collision then they should ultimately be equal to each other.

Similar to energy, momentum might transfer between all sorts of objects, however, their total will remain constant.

Example

The Higgs boson is an elementary particle with "mass"

$$m_H \approx 126\text{GeV} = 6.6 \times 10^{-27}\ \text{kg}$$

It is known to sometimes morph (decay) very quickly into an electron is positron.

$$m_e = m_p = 9 \times 10^{-31} \text{ kg}$$

Now please ignore that the masses dont add up the way you'd think: you can discuss that with Einstein. Our concern here is related to momentum!

Now, suppose be have a Higgs sitting still in our lab somehow. We'd like to know how to arrange our detectors so as to prove it decays into an electron and positron.

COM allows us to predict where the electron and positron should be.

$\underline{before}$

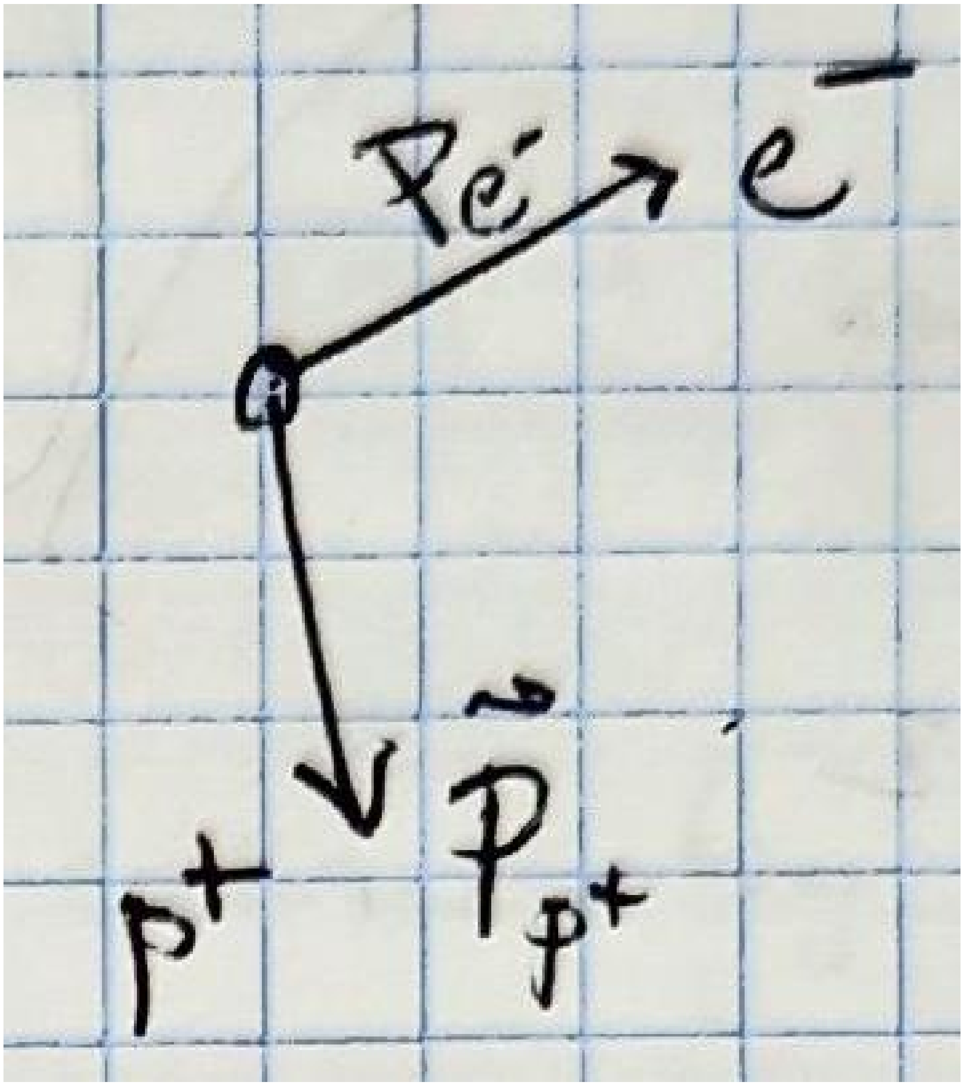
Before the decay the Higgs has no velocity and therefore the total momentum is zero. Easy. Done

$$\vec{p}_i = \vec{0}$$

$\underline{after}$

After the decay (10^{-22} seconds or so) we no longer have a Higgs, but instead have an electron and positron of equal mass. Therefore the total momentum is

$$\vec{p}_f = m_e\vec{v}_e + m_p\vec{v}_p$$



Now conservation of momentum says

$$\vec{p}_i = \vec{p}_f$$

plugging in we have

$$\vec{0} = m_e\vec{v}_e + m_p\vec{v}_p$$

but $m_e = m_p$. Therefore

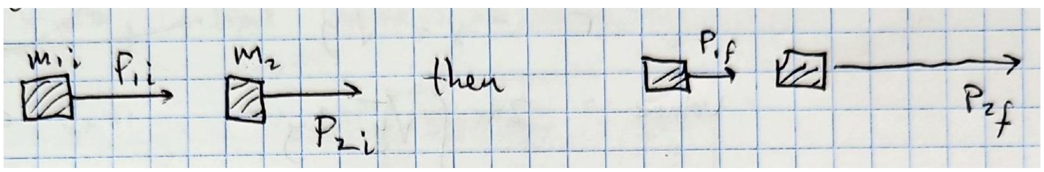
$$\vec{v}_e = -\vec{v}_p$$

So we know that the $p^+$is e^-must go off in exactly opposite directions! We should arrange our detectors diametrically. Furthermore this also p[predicts they should arrive at exactly the same time due to the equal speeds! We can do so much with so little.

There are many other conservations laws that allow us to predict properties of such exotic systems with ease!

Let's dig deeper.

Suppose two cars collide as the figure below: what can we discover?



Now, we know that the change in momentum is caused by the force of the collision and the time over which it occurs.

$$\Delta\vec{p}_1 = \Delta t \cdot \vec{F}_{12}$$

$$\Delta\vec{p}_2 = \Delta t \cdot \vec{F}_{21}$$

$\vec{F}_{12}$ is the force acting on car 2 due to car 1. Similarly $\vec{F}_{21}$ is the force acting on car 1 due to car 2.

Then

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = \Delta t \cdot \vec{F}_{12} + \Delta t \cdot \vec{F}_{21} = 0$$

By Newton's third law $\vec{F}_{12} = -\vec{F}_{21}$

So then this means the total momentum DOESNT change!

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

We call this an "Isolated System". That just means there are no other forces acting. i.e. $\vec{F}_{net} = 0$.

In the above case the normal force and weight cancel and Newton's third law applies to the collision force.

Theory of Collisions

The theory of collisions is an exceedingly important tool in the physicists tool boxThe past 100 years has seen increasing use of this tool in order to understand the fundamental nature of reality The largest and most complex device EVER constructed by humanity is the LHC at CERN in Switzerland. This device is a 27 km long ring of superconducting magnets used to accelerate particles known as hadronse.g. protonsin counter propagating beams at energies in excess of 13 TeV in order to collide them to and create subatomic debris in order to understand the fundamental component of nature and our universeIts greatest discovery was the creation of the Higg's Boson with a mass of 125.35 GeVThis particle is responsible for the mass of electronsquarksand other fundamental particles.

The main idea is to start with initial momenta and measure final momenta as well as all energies. Since energy and momentum (among other things) are conserved then we can understand a lot without much effort.

Here, we begin with more modest ambitions. Let us developthe theory of collisions and add to our problem solving tool boxes.

There are two main types of collisions: elastic and inelastic
elastic collisions.

$$\underline{Perfectly-Elastic}$$

1. Energy is conserved
2. Objects bounce off of each other

$$\underline{Perfectly-Inelastic}$$

1. Energy is not conserved (heat is generated and lost)
2. Objects stick to each other

In the real world we haven't such perfect conditions and collision are generally in-between these two limiting cases. Typically everything is inelastic (although not perfectly inelastic).

Let's begin with perfectly elastic collisions.

Suppose we have 2 objects (atoms, people, cars, cows, etc) which collide and then bounce off in different directions in general. We tally up everything we know before the collision and everything afterwards.

1. Momentum is conserved

$$\vec{p_i} = \vec{p_f}$$

2. Kinetic energy (internal) is conserved.

$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = KE_{initial}$$

$$\frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 = KE_{final}$$

For now we do NOT consider potential energies between the objects or anywhere else.

Let us count the total number of equations we have from these two conservation laws. In 2D the momentum vectors have two components and the energy equation has only scalar values. Therefore, we have three total equations and utilizing these two conservation laws we can, in principle, solve for three unknown quantities. The unknowns may be initial or final momenta, masses, speeds, etc.

Let's write everything out for the following elastic most general 2D perfectly elastic collision. See figure

The masses interact in some way in the shaded blob. we arrint interest in exactly how!

Let us leverage conservation by considering the total energy and momentum before the collision and after the collision

$\underline{Before}$

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = \vec{P}_i$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = E_i$$

After

$$m_1\vec{v}_{1f} + m_2\vec{v}_{2f} = \vec{P}_f$$

$$\frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 = E_f$$

Then we simple set

$$\vec{P}_i = \vec{P}_f$$

and

$$E_i = E_f$$

or one could completely decompose the momenta into x and y components.

We shall analyze this system under a variety of conditions.

First, let us analyze the case of equal masses $m_1 = m_2 = m$ then,

$$m(\vec{v}_{1i} + \vec{v}_{2i}) = m(\vec{v}_{1f} + \vec{v}_{2f})$$

$$\frac{1}{2}mv_{1i}^2 + \frac{1}{2}mv_{2i}^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2$$

Now all the masses cancel and so we have,

$$\vec{v}_{1i} + \vec{v}_{2i} = \vec{v}_{1f} + \vec{v}_{2f}$$

$$v_{1i}^2 + v_{2i}^2 = v_{1f}^2 + v_{2f}^2$$

Typically in an experiment we know initial velocities and can measure the direction (angles) of the momenta.

Second case: unequal masses

Now suppose $m_1 \neq m_2$ but that $\vec{v}_{2i} = 0$.

This is the case that a single moving mass crashes into a different stationary mass and then they both move away in potentially different directions.

Before

$$m_1\vec{v}_{1i} = \vec{P}_i$$

$$\frac{1}{2}m_1 v_{1i}^2 = E_i$$

After

$$m_1\vec{v}_{1f} + m_2\vec{v}_{2f} = \vec{P}_f$$

$$\frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 = E_f$$

Conservation of momentum and energy says:

$$m_1\vec{v}_{1i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$m_1 v_{1i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2$$

Notice the simplification. Initially ALL of the available momentum is in m_1 but afterwards it is distributed between both masses. If we consider that we know the initial data then we have three equations with four unknowns. Unless we know at least ONE of the final velocity components or an angle then we cannot solve this set of equations. In a problem you will typically be given a final speed and/or a deflection angle. Then we would be left with three equations and two or three unknowns which would lead us to a solution/prediction.

Now consider some specific values/data:

$$m_1 = 1 \quad m_2 = 10 \quad \vec{v}_{1i} = v_{ix} = 5$$

Plugging these values into our expressions yields

$$v_{1ix} = 5 = v_{1fx} + 10v_{2fx}$$

$$5^2 = 25 = v_{1f}^2 + 10v_{2f}^2$$

Then we have two equations for two unknowns

$$v_{2fx} = \frac{m_1}{m_2}\left(v_{1i} - v_{1f}\right) = \frac{1}{10}(5 - v_{1fx})$$

Then we can plug this into the energy equation to find

$$25 = v_{1f}^2 + \frac{1}{10}(5 - v_{1f})^2$$

then $v_{1fx} = 5$ or -4.1 m/s

This is entirely reasonable. The value $v_{1f} = 5$ implies that the mass simply passed through the other like a ghost. The -4.1m/s value is the value for the balls actually colliding. Evidently the first mass bounces off and goes the opposite direction. Reasonable.

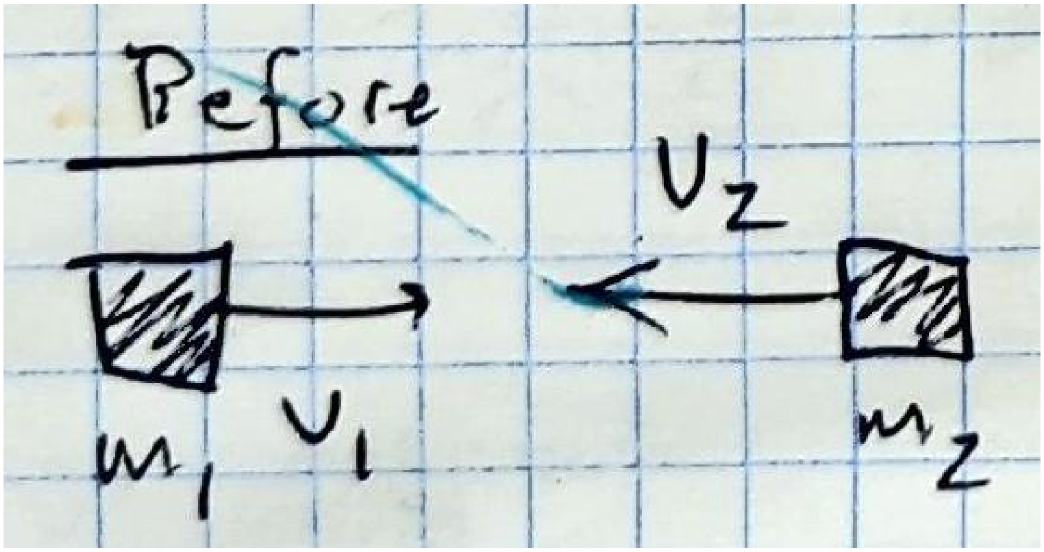
Meanwhile, plugging this in we find that $v_{2f} = 0.91$ m/s. Again, reasonable as the second mass is much more massive than the first, upon collision the second mass moves slowly away in the positive direction.

$\underline{Inelastic - Collisions}$

In inelastic collisions we often have heat and/or we have objects sticking together. In order for two objects to stick together some energy is required and therefore a part of our initial energy goes in to fusing the two or more objects. Therefore a naïve conservation of energy will not work and so in order to leverage COE we will need to keep track of residual heat that we lump together in a term we call Q. In what follows we will consider "perfectly" inelastic collisions (PIC) - collisions that fuse on impact.

Let's begin:

Suppose we have two masses which undergo PIC as below.



We may (aklmost) always leverage COM so let's do so by considering the two moments of "before collision" and "after collision"

BEFORE

as in our figure we have two masses and two speeds moving in opposite directions. Therefore, the total momentum of the system is

$$\vec{P}_i = m_1\vec{v}_{1i} + m_2\vec{v}_{2i}$$

AFTER

After the collision since this is PIC the two masses will be fused as though they are just a single mass $M = m_1 + m_2$ moving at some velocity $\vec{v}_f$. therefore, the final momentum is

$$\vec{P}_f = (m_1 + m_2)\vec{v}_f = M\vec{v}_f$$

Now we can apply COM $\vec{P}_i = \vec{P}_f$
then

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

Therefore, solving for the final velocity we have

$$\vec{v}_f = \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{m_1 + m_2}$$

Now, nothing stops us from looking at the before and after kinetic energies.

$$KE_i = \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2$$
$$KE_f = \frac{1}{2}\left(m_1 + m_2\right) v_f^2$$

Then, the ratio of these energies will give us a kind of "efficiency" in that we know $KE_f < KE_i$ since the objects fuse. So then we have

$$\frac{KE_f}{KE_i} = \frac{\frac{1}{2}\left(m_1 + m_2\right)\left[\frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}\right]^2}{\frac{1}{2}\left[m_1 v_{1i}^2 + m_2 v_{2i}^2\right]}$$

cleaning up

$$\frac{KE_f}{KE_i} = \frac{m_1^2 v_{1i}^2 + m_2^2 v_{2i}^2 + 2m_1 m_2 v_{1i} v_{2i}}{\left(m_1^2 + m_1 m_2\right) v_{1i}^2 + \left(m_2^2 + m_1 m_2\right) v_{2i}^2}$$

It's admittedly a mess but it's only algebra. Now let's apply some assumption in order to simplify this.

Suppose m_2 is initially at rest. Then, plugging in 0 for $\vec{v_{2i}}$ we have

$$\frac{KE_f}{KE_i} = \frac{m_1}{m_1 + m_2}$$

The fraction of kinetic energy lost in this PIC is then

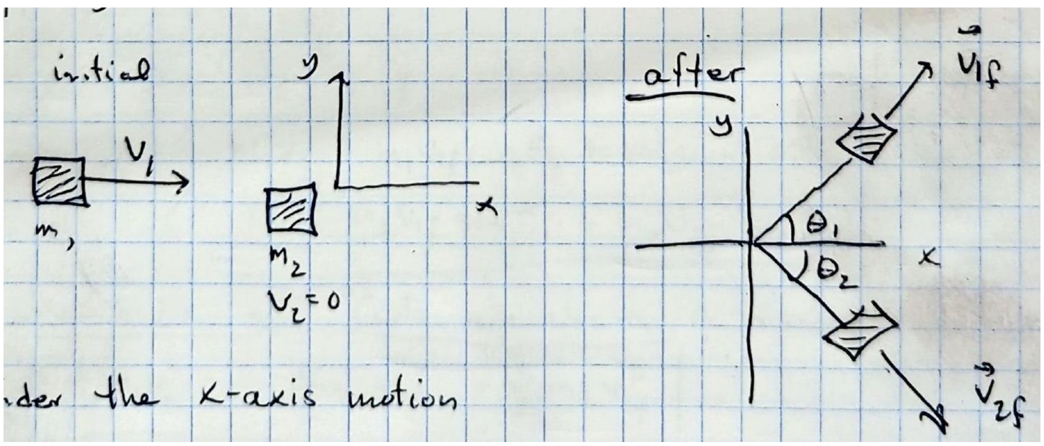
$$\frac{KE_i - KE_f}{KE_i} = \frac{m_2}{m_1 + m_2}$$

Typically PIC is easier than PEC since in PEC we have three equations three unknowns. In PIC we only use COM to find v_f which is often quick and easy!

General 2D Collisions

In 2D everything is the same but now we have to decompose vectors explicitly.

Suppose, a mass m_1 collides perfectly elastically with mass m_2 at rest such that the collision is slightly off center. Imagine billiard balls for definiteness. In this case we expect the two balls to fly off at different angles with respect to the site of the collision. See the figure below



To proceed we must decompose the momentum/velocity vectors as follows. Choose the coordinates in the figure above:

Initially we have the velocity of m_1 one to be

$$\vec{v}_{1i} = \begin{bmatrix} v_{1x} \\ 0 \end{bmatrix}$$

the velocity of m_2 is zero.

The tricky part is the final momenta. Suppose after the collision the two mass go off with angles θ_1 and θ_2 as in the figure above. Then we may decompose the 2D vectors as

$$\vec{p}_{1f} = \begin{bmatrix} p_{1f}\cos\theta_1 \\ p_{1f}\sin\theta_1 \end{bmatrix}$$

and

$$\vec{p}_{2f} = \begin{bmatrix} p_{2f}\cos\theta_1 \\ -p_{2f}\sin\theta_1 \end{bmatrix}$$

Be very careful about the minus sign in the decomposition!

Now we can demand conservation of momentum as

$$\vec{P}_i = m_1\vec{v}_{1i}$$

$$\vec{P}_f = \vec{p}_{1f} + \vec{p}_{2f}$$

or,

$$m_1\vec{v}_{1i} = \vec{p}_{1f} + \vec{p}_{2f}$$

Writing it out completely we have

$$m_1\begin{bmatrix} v_{1i} \\ 0 \end{bmatrix} = m_1\begin{bmatrix} v_{1f}\cos\theta_1 \\ v_{1f}\sin\theta_1 \end{bmatrix} + m_2\begin{bmatrix} v_{2f}\cos\theta_1 \\ -v_{2f}\sin\theta_1 \end{bmatrix}$$

Notice that we have two equations here and four unknown final velocity components. The x components and the y components.
Therefore we have

$$m_1v_{1i} = m_1v_{1f}\cos\theta_1 + m_2v_{2f}\cos\theta_2$$

and

$$m_1v_{1f}\sin\theta_1 = m_2v_{2f}\sin\theta_2$$

Thats the best we can do with the given information. However we could still impose conservation of energy in order to narrow down a solution.

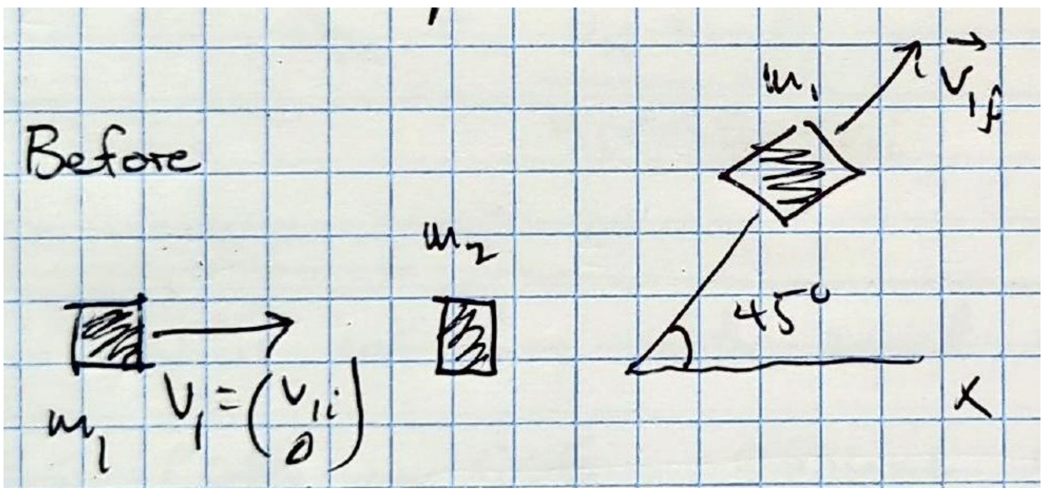
$$\frac{1}{2}m_1v_{1i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

Now counting the number of equations and number of unknowns we find three equations and four unknowns, viz, $v_{1f}, v_{2f}, \theta_1, \theta_2$

In order to make headway an experiment would need to measure at least one of these unknowns. Usually we measure a scattering angle θ_1 or θ_2

Example

Suppose we have some scattering process where we are only able to keep track of m_1. However we know the mass m_2 and its velocity is initially 0 . m_1 is sent along x-direction and scatters into a 45° angle. We ask the question: what does m_2 do? see figure below



Well, this problem is VERY similar to all the hard general work we already did analyzing this type of situation so we can use the equations we developed already. Here we are given θ_1 and so our three equations become solvable!

Let's see,

$$m_1 v_{1i} = m_1 v_{1f} \cos 45 + m_2 v_{2f} \cos \theta_2$$

then we can solve for $v_{2f} \cos \theta_2$

$$\frac{m_1 v_{1i} - m_1 v_{1f} \sqrt{2}/2}{m_2} = v_{2f} \cos \theta_2$$

But, from our other equation we have

$$0 = m_1 v_{1f} \sin 45 + m_2 v_{2f} \sin \theta_2$$

which yields

$$v_{2f}\sin\theta_2 = -\frac{m_1}{m_2}v_{1f}\sin 45$$

Next, we can divide our two equations to find

$$\tan\theta_2 = \frac{v_{1f}\sin 45}{v_{1f}\cos 45 - v_{1i}}$$

or for a general scattering angle θ_1

$$\tan\theta_2 = \frac{v_{1f}\sin\theta_1}{v_{1f}\cos\theta_1 - v_{1i}}$$

Now lets look at a PEC with equal masses. Consider an experiment whereby we fire a proton at another proton at rest. We anticipate they do not collide perfectly in line so we might assume some non-zero deflection angles

Let's assume assume m_2 is at rest and we fire a proton at $\vec{v}_{1i}$ along the x-axis

COM

$$\begin{aligned} m\vec{v}_{1i} &= m\vec{v}_{1f} + m\vec{v}_{2f} \\ &= \begin{cases} mv_{1i} = mv_{1f}\cos\theta_1 + mv_{2f}\cos\theta_2 \\ \quad 0 = mv_{1f}\sin\theta_1 + mv_{2f}\sin\theta_2 \end{cases} \end{aligned}$$

COE

$$\frac{1}{2}mv_{1i}^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2$$

We see the mass drops out of all of the equations.

$$\begin{aligned} \therefore \quad v_{1i} &= v_{1f}\cos\theta_1 + v_{2f}\cos\theta_2 \\ 0 &= v_{1f}\sin\theta_1 + v_{2f}\sin\theta_2 \end{aligned}$$

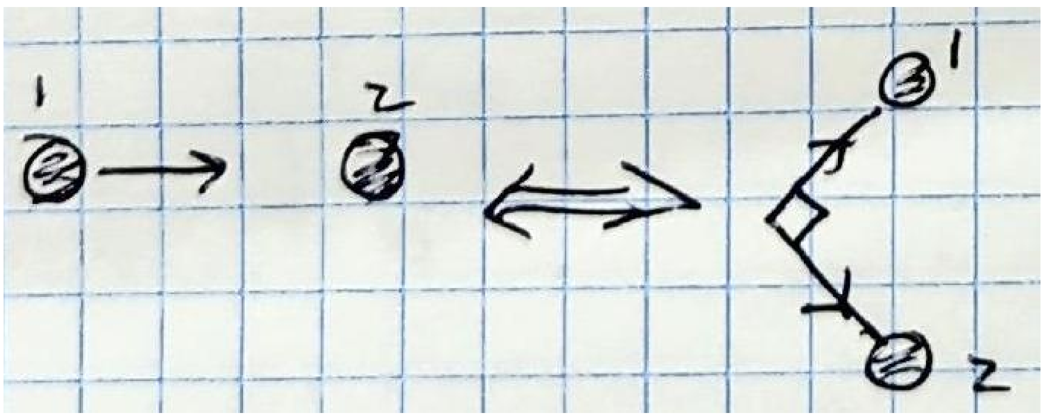
and

$$(v_{1f} \cos\theta_1 + v_{2f} \cos\theta_2)^2 = v_{1f}^2 + v_{2f}^2$$

You may verify that these equations are be solved by

$$v_{1f} v_{2f} \cos(\theta_1 - \theta_2) = 0$$

We find that if $\theta_1 - \theta_2$ is 90° then this is true.



Or if if $v_{1f} = 0$ or if $v_{2f} = 0$ this is true.

Therefore, the two masses MUST take perpendicular trajectories and in three dimensions would would anticipate an entire right angle cone about the x-axis.

Torque and Equilibrium

Perhaps when you first learned about forces you wondered why we didnt discuss "where" the forces are applied. Surely this is of utmost importance in the universe, right? Push on the center of a stick is MUCH different than pushing on the top or anywhere else. So why did we ignore such a MAJOR thing?

Well, like all things in physics, we have to start with the absolute simplest situations and spend time and practice building ourselves up to more complicated features of verifiable reality.

The studies we are about to embark on are fantastically complex. We will see incredible phenomena that we could have never imagined. Absolutely baffling phenomena will confound us and we will come to realize that we must lean on the mathematics in order to infer our intuitions. On top of that we must start working in full 3D space. Although we will take a step into the z direction we shall do it in the simplest possible manner by considering vectors that are entirely perpendicular to the (x, y) plane. The full generality is but a hop and a skip from there.

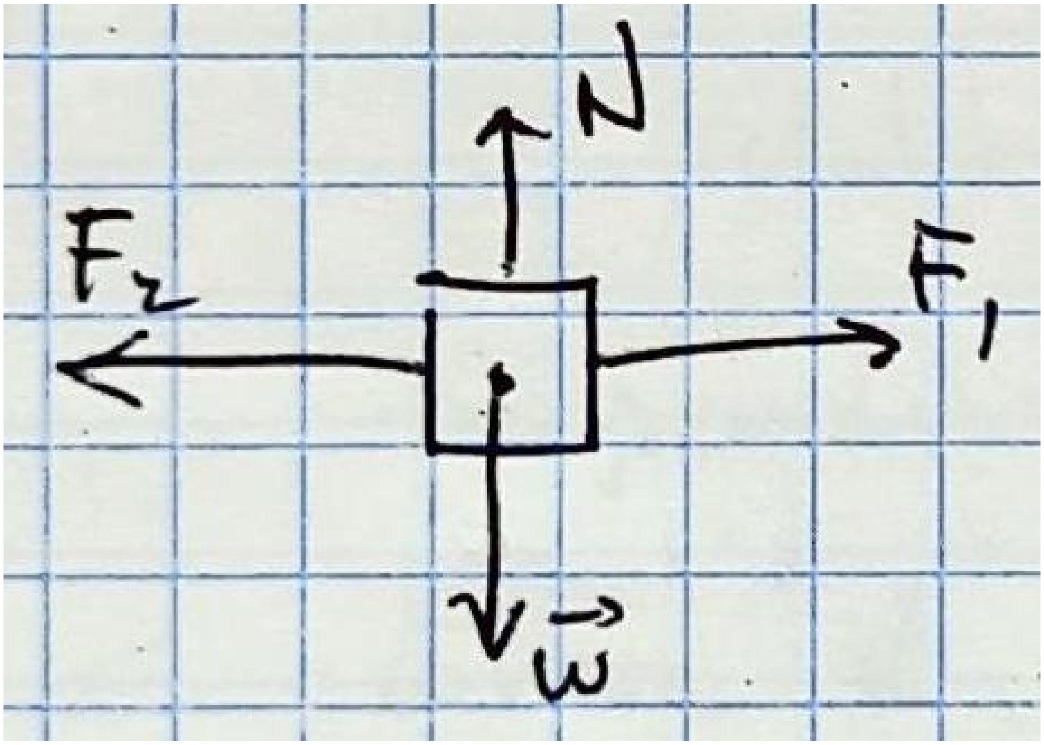
To get things started let us recall equilibrium for mechanical systems. Equilibrium means $F_{\text{net}} = 0$. There way be many non-zero forces acting on an object but when sum them all up the will precisely cancel and add to zero which means the object is either at rest or moving at constant velocity.

If $\sum \vec{F}_i = 0$ then we must have $\sum F_x = 0$ and $\sum F_y = 0$.

We define two types of mechanical equilibrium:

1. Static: $v = 0$
2. Dynamic: $v \neq 0 = \text{constant}$

For example, see below

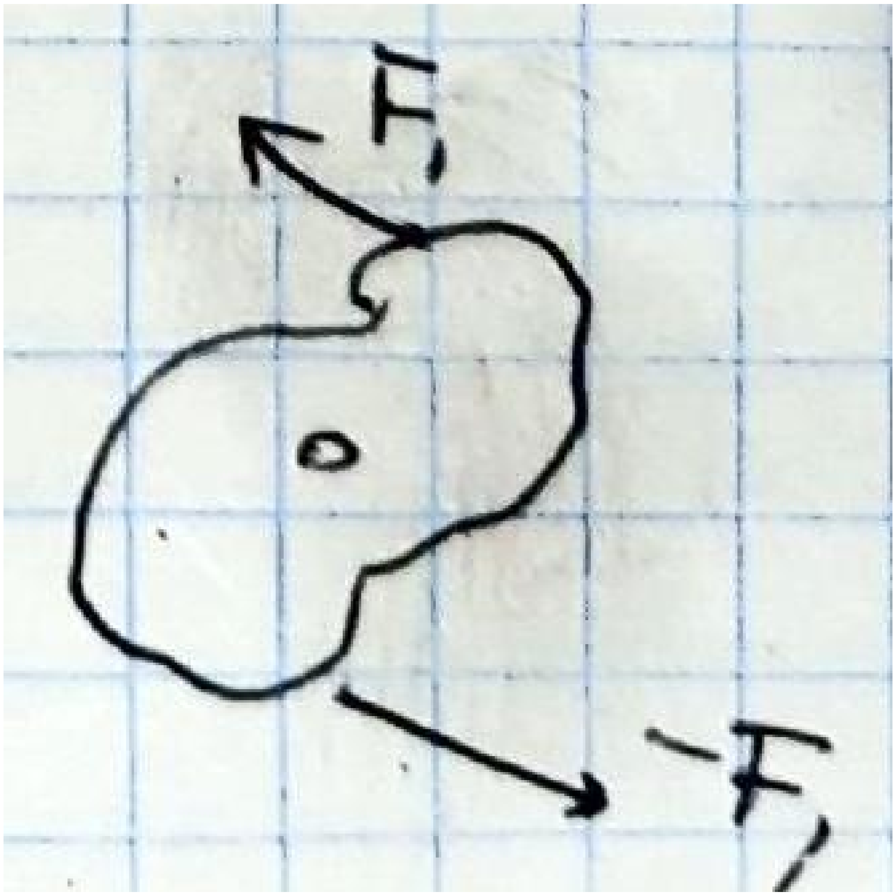


$N = -\vec{w}$ and $F_1 = -F_2$

However, we were just discussing how this is not the whole story. It may be that an object is spinning. Recall that each mass moving in a circle experiences a centripetal force

$$a_c = \frac{v^2}{R}$$

If we have a potato and push on the top with F_1 and the bottom with $-F_1$, then $\sum F = 0$ although there still exists a centripetal acceleration for each little piece of potato!



So if seems we must take account of WHERE the forces are applied. The net force adding to zero is not enough to establish equilibrium. We need to figure out how to say "it wont spin faster ands faster". This leads to the notion of "torque".

If a mechanical system is in equilibrium then we may also allow for constant angular velocity i.e. we need the angular acceleration to also be zero, i.e. $\alpha = 0$.

Now we have two conditions for mechanical equilibrium

1. there are no linear accelerations

2. there are no angular accelerations

The first says that the forces add to zero, but what of the second?

Let's make a definition:

Torque: A torque is a force applied to an object at a specific distance from a fixed point. It can be thought of as a "rotational force". Torque ($\vec{\tau}$) is a vector quantity whos magnitude is given by

$$|\vec{\tau}| = |\vec{F}||\vec{d}|\sin\theta$$

The complete definition for torque is based upon the "cross product" of two vectors as:

$$\vec{\tau} = \vec{F} \times \vec{r}$$

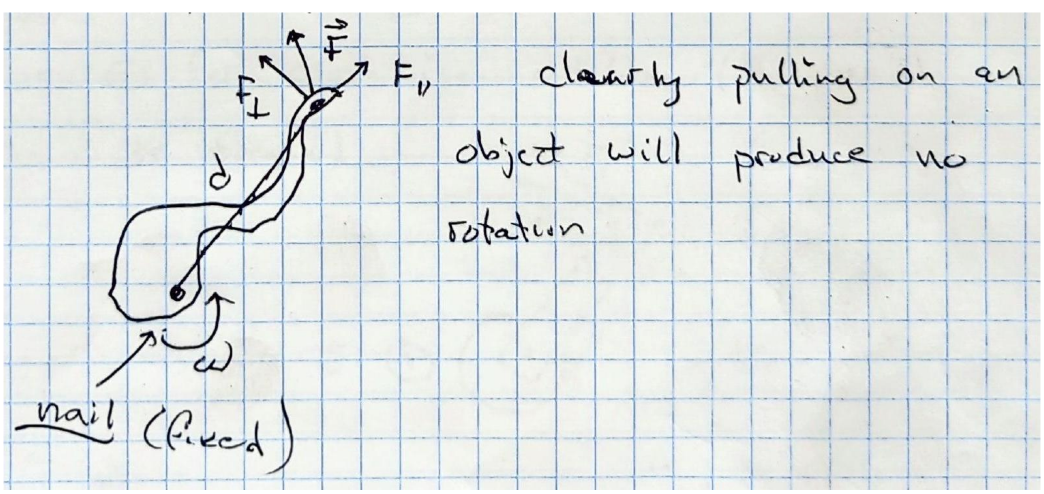
where the cross product $\times$ can be defined by the formula above for the magnitude and the rule that the cross product is ALWAYS perpendicular to the other two vectors. In our studies the positions and forces will lie in the (x, y) plane so that all torques will be pointing along the vertical z-axis.

There are other expressions for the cross product5 that are useful but which we will not need in these studies.

Importantly

$$\vec{\tau} = \vec{F} \times \vec{r} = -\vec{r} \times \vec{F}$$

This equation $|\vec{\tau}| = |\vec{F}||\vec{d}|\sin\theta$ says that a torque (a rotational force) is equal to the perpendicular component of $\vec{F}$ times where it acts in relation to a fixed rotational point which we call the **pivot** P



The longer/farther from the pivot the force acts, then the greater will be the torque.

Now, torque is a vector with units of

$$[\tau] = N \cdot m$$

But recall that this is our definition of what energy units are! Dear reader, it is VERY confusing indeed but nobody ever says a torque is some amount of energy. Energy is scalar and torque is vector and the objects have nothing to do with each other. When we talk about some amount of torque we will always say "Newton meters" or "foot pounds". They just happen to have the same base units but they are fundamentally different beasts.

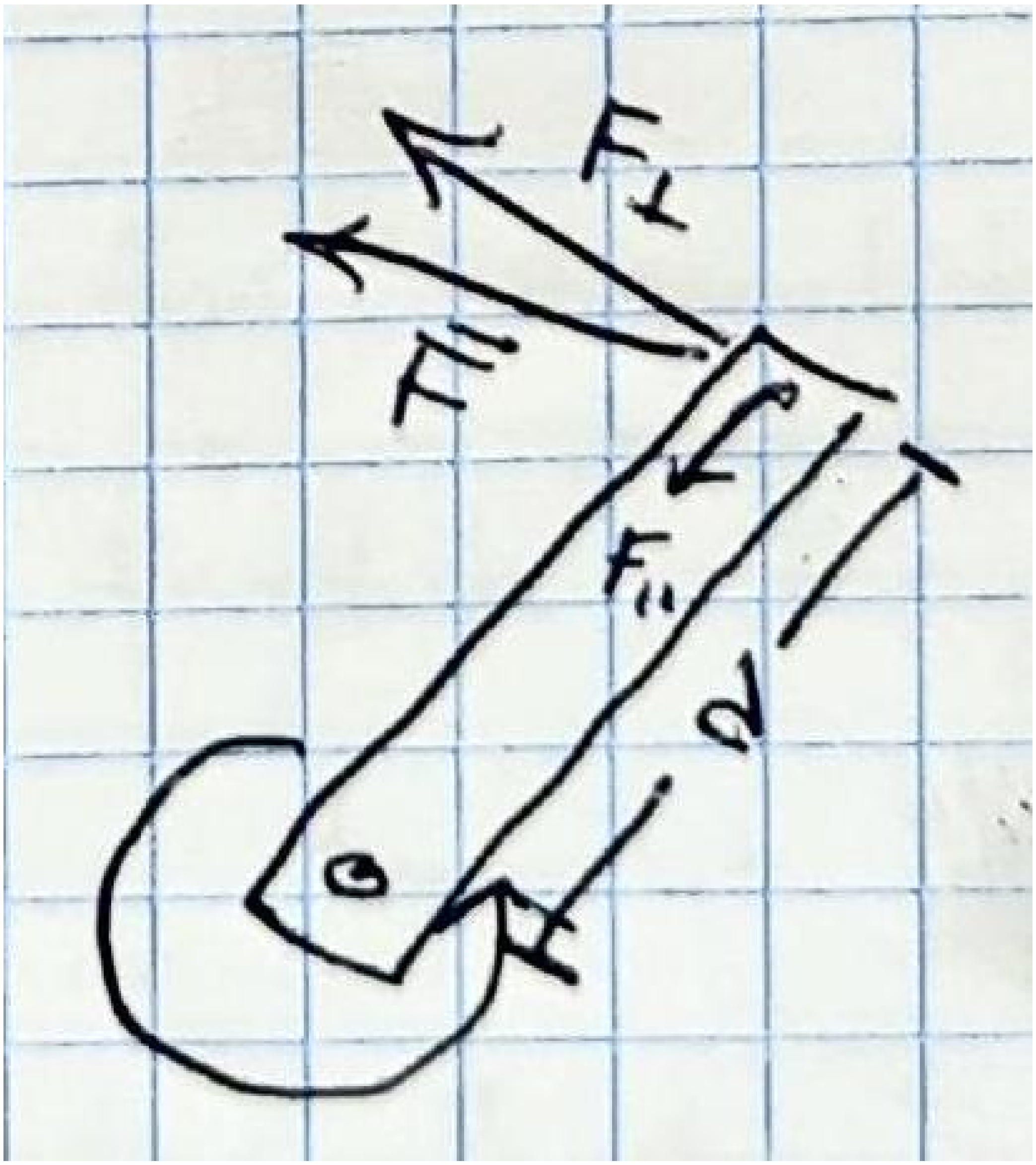
Now, we're welcome to write our magnitude equation a couple different ways that will help us geometrically.

1. $\tau = (F_\perp)$ r
2. $\tau = (F)r_\perp$

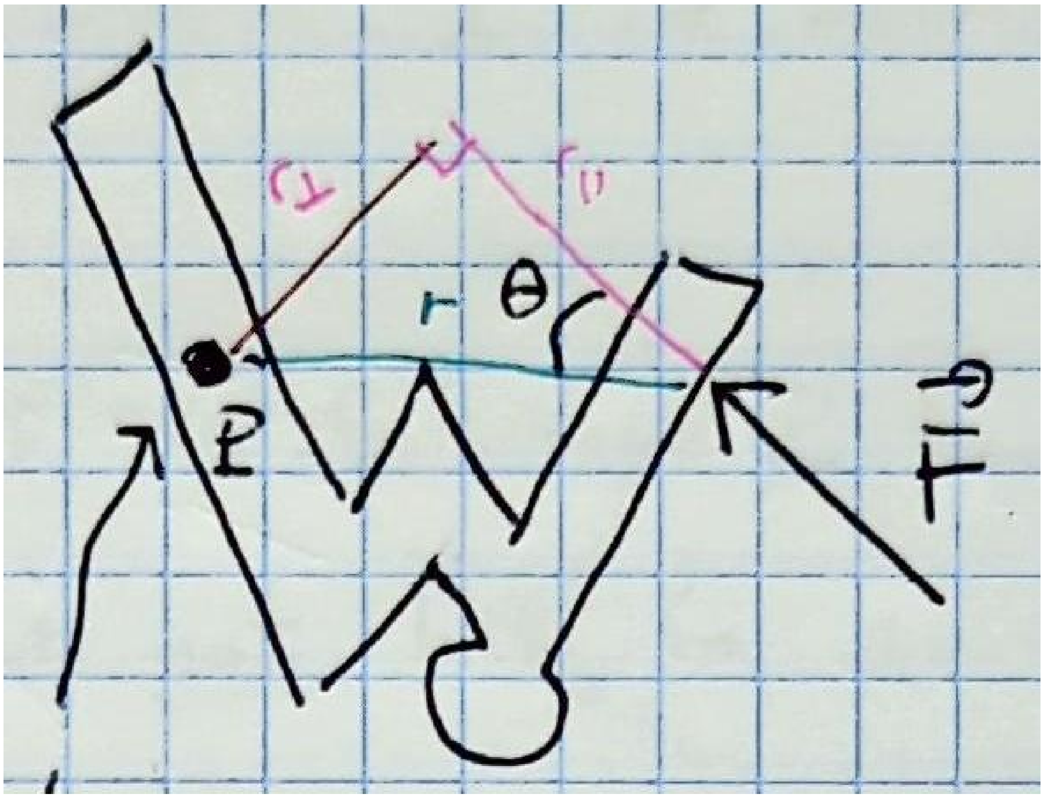
where $F \sin\theta = F_\perp$ and similarly for the position vector. These formula say that 1) you can consider only the perpendicular component of the full force to be acting a distance r from the pivot. or 2) you may consider the full force acting a distance $r_\perp$ from the pivot.

It's cleaner if you inspect the pictures below.

1.



2.



3.

Now, if there exists a torque on a system then that system will accelerate about its pivot point. In order for $\alpha = 0$ and $\omega =$ constant then we must have that all torques sum to zero.

$$\sum_i \vec{\tau}_i = \vec{\tau}_{net} = 0$$

This is our second condition for mechanical equilibrium.

Torques can either positive or negative. We define a positive torque to be a counter clockwise rotation and a negative torque will signify a clockwise rotation. (this is due to the vector nature of torque). Generally, you will be finding the magnitude of the torque using the above formula and then imposing the direction to be $\pm$.

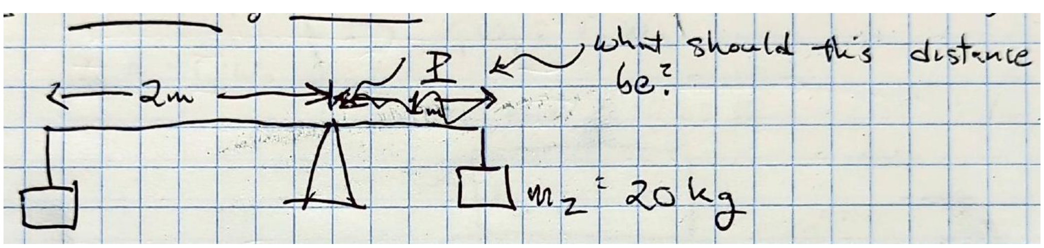
In all of the following peroblems we face covering mechanical equilibrium we will require two conditions

1. $\sum \vec{F}_i = 0$
2. $\sum \vec{\tau}_i = 0$

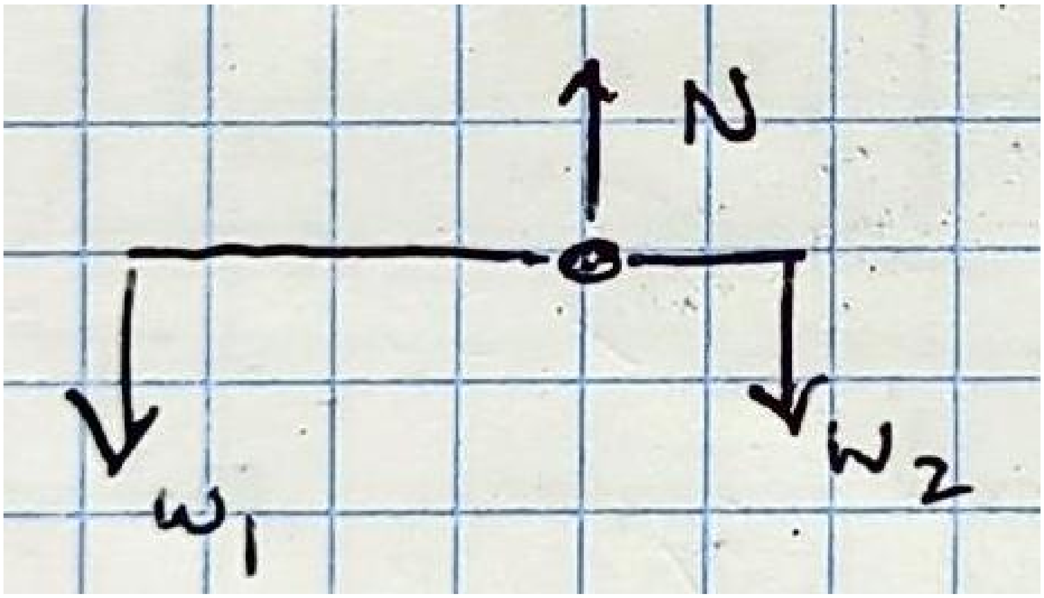
We always calculate torque with respect to some pivot point. The object may not actually be pivoting about this point! It is OUR choice to call any point the "pivot" much like we could choose any coordinate system we please. Our equations apply to ANY pivot point: the object is in EQ or it's not. That's not to say we shouldn't be judicious in our choice of P. Almost always there will be a practical point to consider that will simplify the amount of work we must do to solve a problem. However, once we choose our pivot point we must use ONLY that point: there can be only one. We will see some examples later that show how this works!

Example: Balancing Masses

In the following we assume the bar has zero mass. $m_1 = 10$ $m_2 = 20$ $r_1 = 2$ and r_2 is to be determined. See diagram below



1. As usual we do our strategy - draw, FBD, coordinates, equations, solve. What is FBD for this system? On the FBD we now keep track of where the forces are and we do not label torques.



2. What is torque produced by each force? Each force will produce a torque about the pivot point WE choose. Here we shall choose the natural pivot point to be precisely where the normal force is occurring.

Recall we say $+\tau$ if rotation is CCW and $-\tau$ if rotation is CW
Let's consider the first weight $\vec{w}_1$.

$\vec{w}_1$ is producing a torque about the pivot in the CCW sense. Therefore it will be a positive torque. It acts at a distance r_1 from the p[pivot at a 90° angle. Therefore,

$$\vec{\tau}_1 = +r_1 w_1 = +m_1 g r_1 = 200$$

Now for $\vec{w}_2$. First we must determine the sign of the torque about the pivot due to $\vec{w}_2$. In this case the rotation is CW and so we find a negative torque

$$\vec{\tau}_2 = -r_2 w_2 = +m_2 g r_2 = -200 r_2$$

Last we MUST consider the normal force. It's magnitude is equal to the sum of the weights (otherwise the whole thing would float away or something.

That is to say our first EQ condition determines $\vec{N}$

1. $\sum \vec{F}_i = 0 = -m_1 g - m_2 g + N$

or $N = m_1 g + m_2 g$
Our second EQ condition will allow us to solve for r_2

2. Solve $\sum \tau_i = 0$

Well, $\vec{\tau}_1 + \vec{\tau}_2 = 0 \Rightarrow \vec{\tau}_1 = -\vec{\tau}_2$
But we found these

$$\vec{\tau}_1 + \vec{\tau}_2 = 200 - 200 r_2$$

and so we find $r_1 = 1$
In general, for two balanced masses

$$r_2 = \frac{m_1}{m_2} r_1$$

Now, we could imagine much more complicated balance beams. If we have 3 or more weights we just have more torques acting about a pivot P. e.g.

$$\left.\begin{array}{l} \sum \tau_i = r_1 w_1 + r_2 w_2 - r_3 w_3 = 0 \\ \sum F_i = w_1 + w_2 + w_3 - N = 0 \end{array}\right\}$$

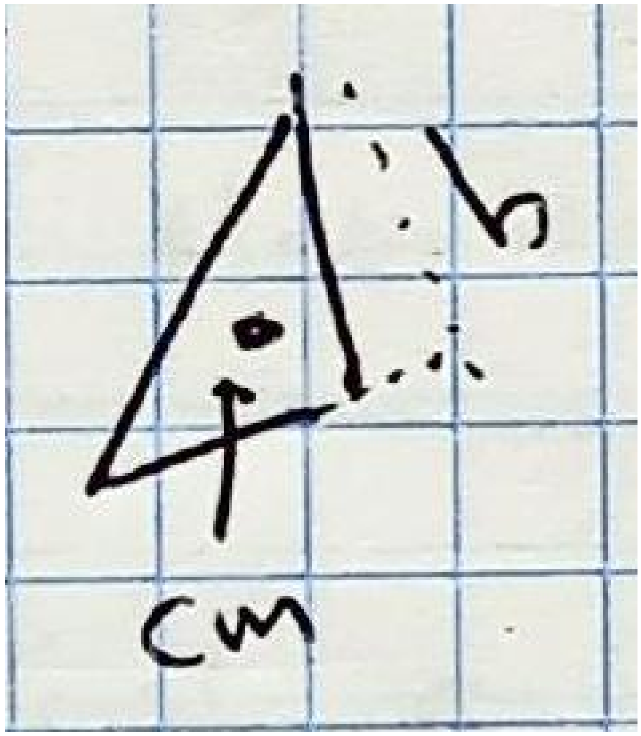
Center of Mass/Gravity

The center of mass (CM) is a single point that we can imagine all of the mass of a body lies. This is usually based on symmetry but complex formulas exist for arbitrary shapes.

For example, if the total mass is evenly distributed then we have

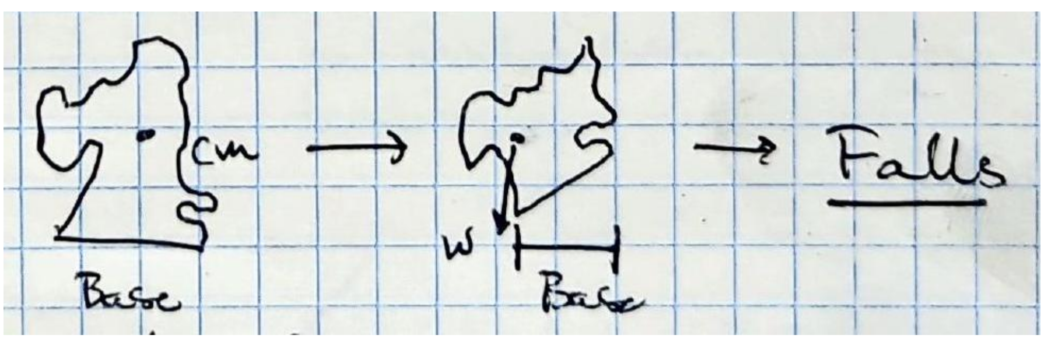
Sphere: All of the mass of a sphere can be considered to be at the center of the sphere.

Triangle: the CM is at $\frac{h}{3}$ where h is defined in the figure below



For a rectangle the CM is at $h/2$

Theorem: A body will fall over when the CM tips over the base of support.



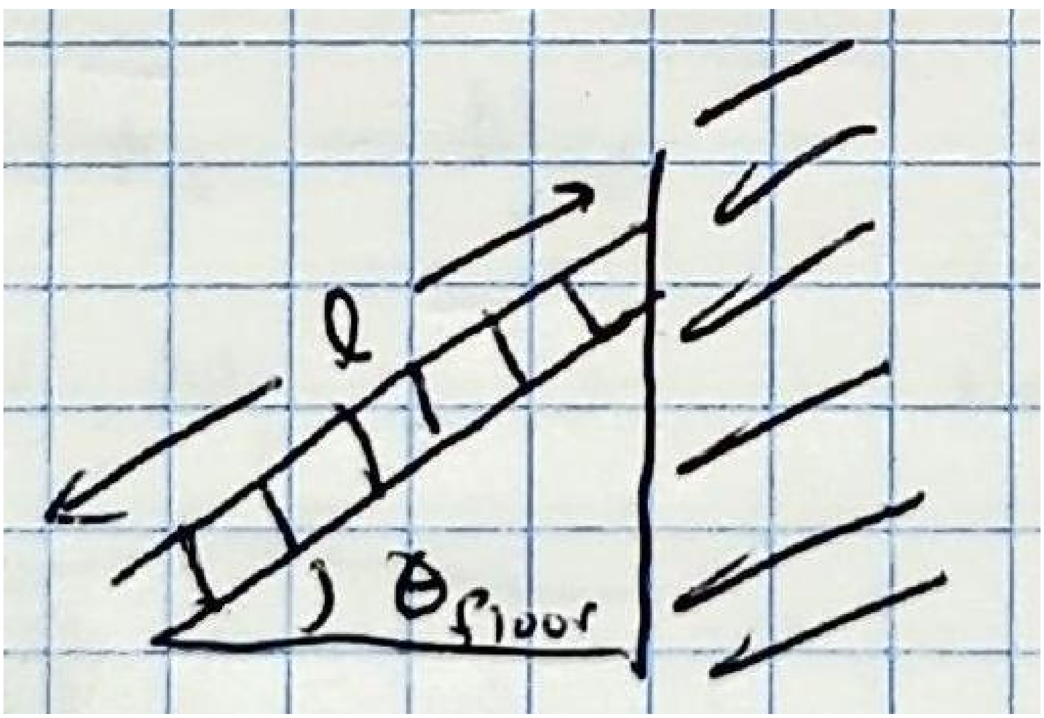
For example, balancing your pen on its point is very unstable. A slight breeze (or any infinitely small perturbation) will cause the CM to tip over its base (which is itself a very sharp point).

Skyscrapers have very large bases and lower CM so that they can remain standing even under high winds and earthquakes. If a skyscraper had the majority of its mass on the top floor thenb it would require very little deflection to bend that CM over the base.

Conversely, and inflatable toy with sand contained at the based a centimeter above the pivot is impossible to knock over!

Example: A ladder against a wall

We wish to find the normal forces and coefficient of friction in the figure below. We consider the wall to be frictionless but the floor shall be carpeted with some coefficient of friction. The angle at the floor shall be 53° and the weight of the ladder is 400N



The only tools at are disposal are our two conditions for mechanical equilibrium.

Our strategy is to find all torques about a pivot that we choose and sum them to zero. We will also add up all forces and set them to zero.

Once we have our values plugged in we will try to see if we can solve for the friction and normal forces.

We have a quite complicated geometry so we shall proceed slowly.

Step 1:

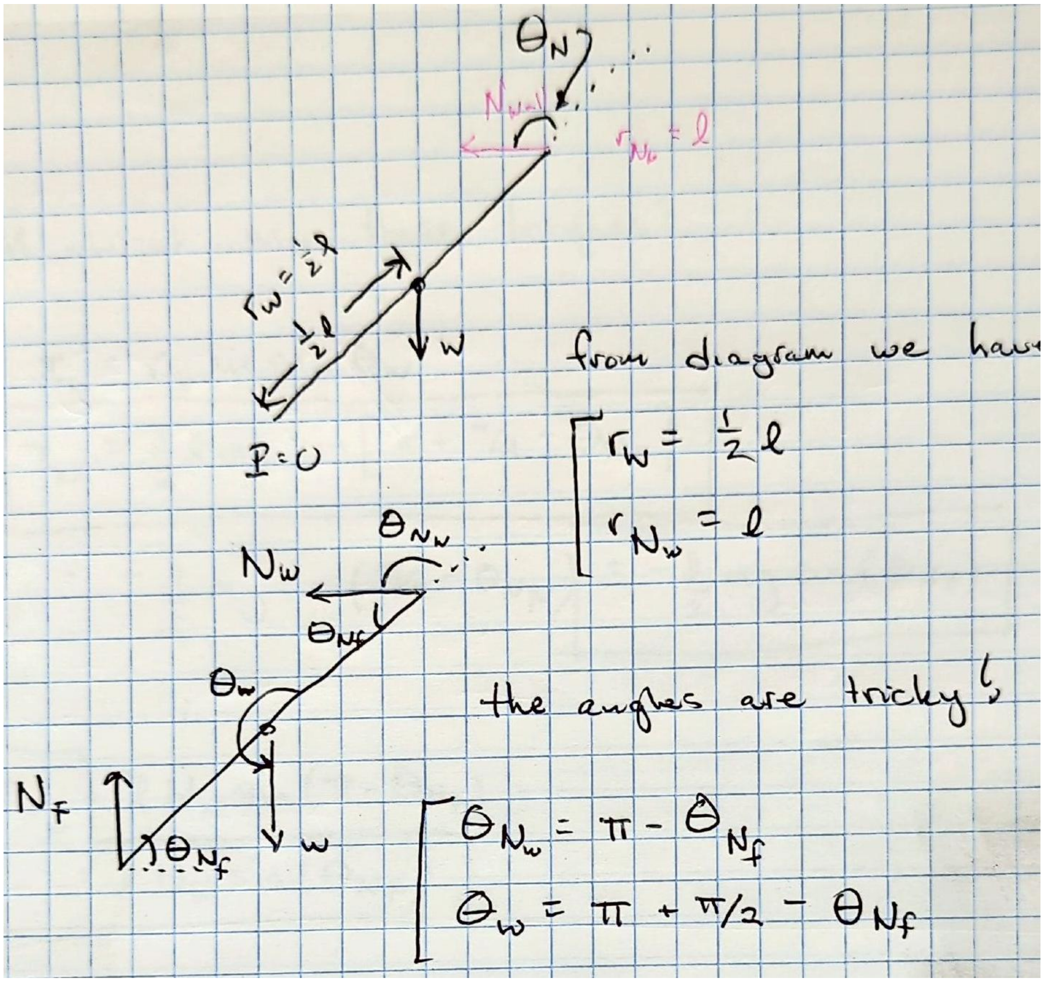
The ladder has weight $\vec{W}$ and length l

We want to find Normal (reaction) forces due to the wall and the floor.

What are all the forces?

$$\begin{cases} \vec{N}_{floor} \\ \vec{f_\mu} = \mu_s N \\ \vec{w} = -mg \\ \vec{N}_{wall} \end{cases}$$

See diagram below for the directions.



Now that we have all of our forces identified we should next define where we would like to consider our pivot point P. We shall freely choose P to be the contact point of the ladder with the floor and also define this as the origin of our coordinates.

The, from the diagram we have

$$r_w = \frac{1}{2}l$$

is the distance from the pivot to the weight vector (viz, the weight is at the CM)

Next we have3 the distance to the wall normal force

$$r_{N_w} = l$$

and the floor's normal is acting precisely at the pivot so that its distance equals zero.

Now, the angles are MUCH trickier. From our diagram we must find the angle between each force and the ladder

$$\theta_{N_w} = \pi - \theta_{N_f}$$

$$\theta_m = \pi + \frac{\pi}{2} - \theta_{N_f}$$

where θ_m is the angle between the ladder and the weight vector.

Now we are ready to apply our EQ conditions

1. $\sum \vec{F}_i = 0$

$$\sum F_x = f_\mu - N_w$$

$$\sum F_y = N_F - w$$

which implies that

$$f_\mu = N_w \qquad N_F = w$$

Now for the our torque condition:

2. $\sum \vec{\tau} = 0$

About $P = 0$ we have

$$\tau_m + \tau_{N_{wall}} = 0$$

Since we chose P to be at the floor then the friction and the floor's normal produce no torque! Generally you often want to pick P to be a point where as many forces are at as possible. Here we killed off two by picking that point!

Okay, now what are these torques?

$$\tau_m = r_m w \sin\theta_m == \frac{1}{2} lmg \sin\left[\pi + \pi/2 - \theta_{N_F}\right]$$

$$\tau_{N_w} = lN_w \sin\left(\pi - \theta_{N_F}\right)$$

so that

$$\sum \tau = l\left[N_{wall}\sin\theta_{N_f} - \frac{1}{2}mg\cos\theta_{N_f}\right] = 0$$

$$\theta_{N_f} = 53^\circ \qquad w = 400$$

So that we find

$$N_{wall} = 150.7$$

and

$$\mu_s = 0.377$$

Finally the net force AT $P = 0$ is

$$\begin{aligned}\vec{F} &= \vec{f} + \vec{N} = \begin{bmatrix} +150.7 \\ +400 \end{bmatrix} \\ |\vec{f} + \vec{N}| &= 427.4 \text{ N} \\ \phi &= \tan^{-1}\left(\frac{N}{f}\right) = 69.3^\circ\end{aligned}$$

Note that

1. We didn't consider $\pm\tau$ (directions) this all popped out of our choices of angles.
2. Nothing is actually rotating
3. The length l didn't matter one lick

Simple Machines

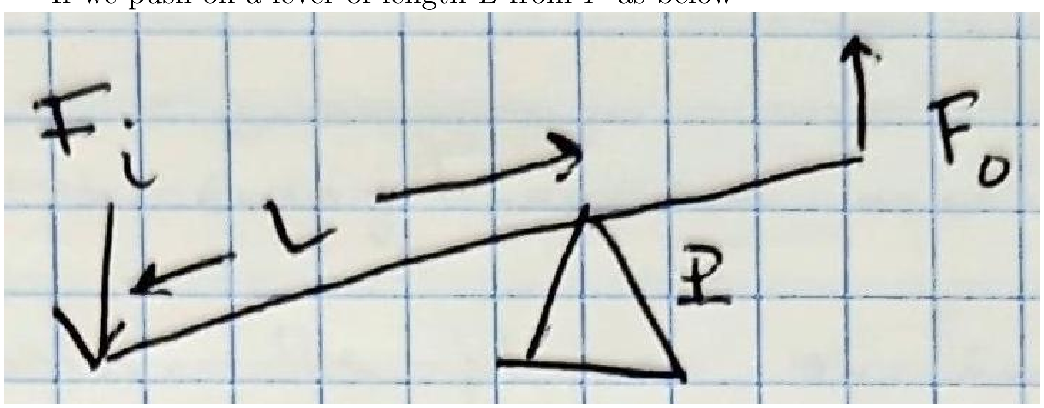
Levers, Gears, Pulleys, cranks, etc'

Let us define the mechanical advantage (MA) of a system

$$MA := \frac{F_{\text{output}}}{F_{\text{input}}} = \frac{F_0}{F_i}$$

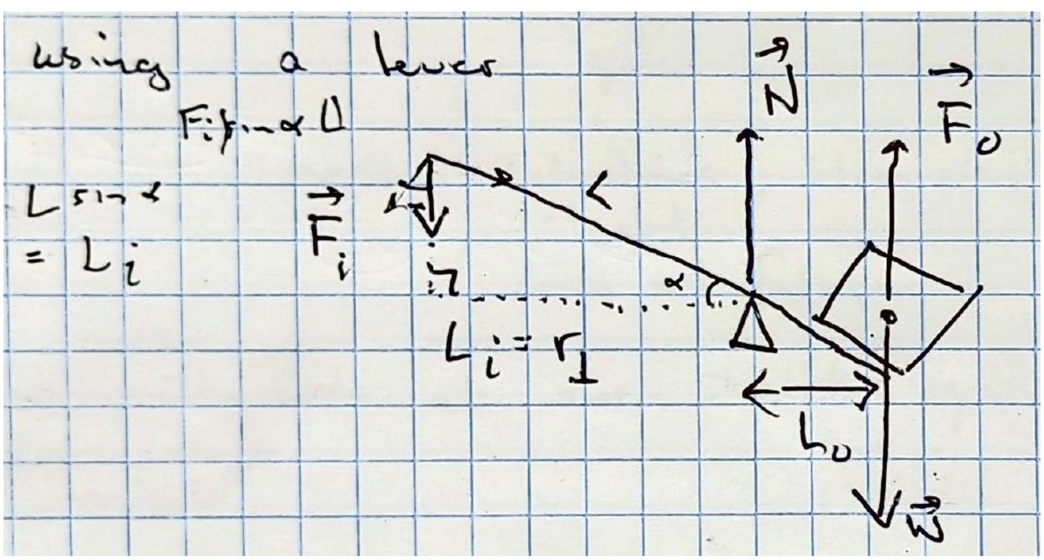
We can utilize torques and pivots to amplify forces over some distance as follows.

If we push on a lever of length L from P as below



then we can lift quite heavy objects (F_0)

Suppose we want to lift a massive Egyptian pyramid stone as below



Then we have

$$\sum \tau_i = 0 = -F_i L_i + F_0 L_0$$

where $\vec{F_i}$ is the force we "input" and $\vec{F_0}$ is the output force.
Therefore we have

$$MA = \frac{F_0}{F_i} = \frac{L_i}{L_0}$$

This implies we can lift exceedingly heavy weights at the expense of increasing our lever's length. At some point you'll have a lever that's so long that a slight input force will snap it in half.

Bio-Mechanics

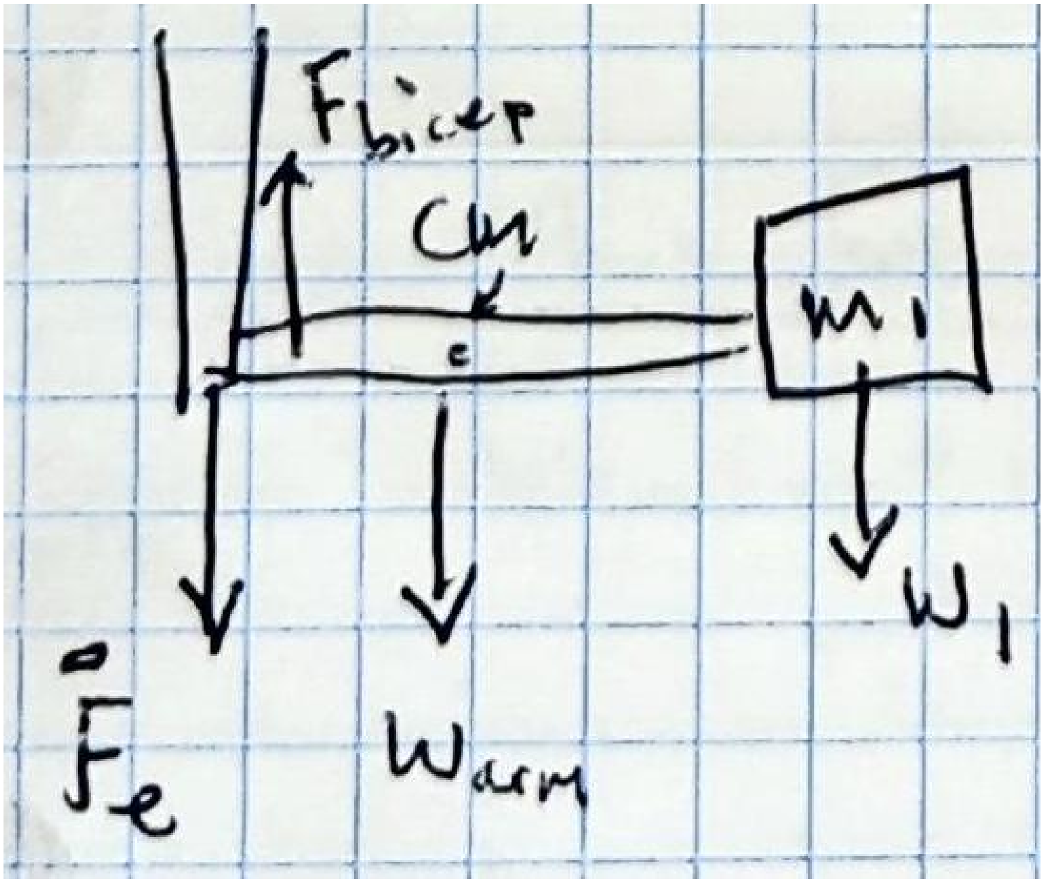
We now have everything we need to understand basic bio-mechanics: the physics of such things as walking, bending. lifting, stooping, etc.

We have two types of muscles; skeletal and smooth. Skeletal muscles are attached to bones and tendons. They exhibit rapid contraction/relaxation and becomes fatigued very quickly.

Smooth muscles are those found in the intestines, stomach, etc. The muscles operate more slowly and do not (thankfully!) fatigue.

We will look at our skeletal system and ignore any smooth muscle.

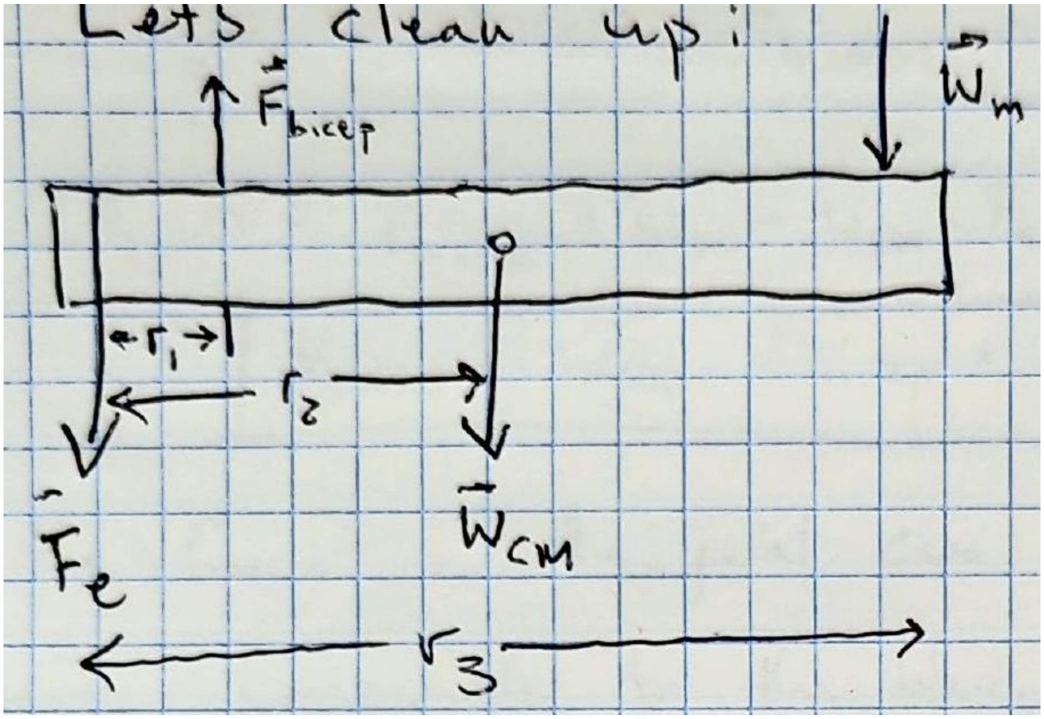
Suppose we lift a weight using primarily our biceps and we hold it for an extended time. What forces are at play? We sketch our system in as simple of a way as possible (those readers with better artistic skills are always encouraged to make their physics homework beautiful). See below



These are the four primary forces at play (there may be more - for example abdomen, back, legs, etc- but we wish to keep it simple). If we know where they act then we can use our two rules

$$\sum F_i = 0 \quad \sum \tau_i = 0$$

Let's clean up our drawing for further simplification and label our pivot and distances. We have the force on the elbow (which is our chosen pivot), the force due to the bicep, the weight of the arm, and finally the weight we're holding:



1. $\sum \tau_i = 0$

The torque due to F_e (elbow) is zero since this is our pivot P
Then the rest are

$$\tau_{bicep} = F_{bicep} r_1$$
$$\tau_{arm} = -w_{arm} r_2$$
$$\tau_m = -w_m r_3$$

All of these forces act perpendicular to the arm so that $\theta = \pi/2$
We can solve this for the bicep force

$$F_{\text{bicep}} = \frac{w_{arm} r_2 + w_m r_3}{r_1}$$

If we knew the values then we can find the force our biceps apply. Typical values are $400 - 1000$N

Since we found F_{bicep} we can solve for F_{elbow}

$$\sum F_i = 0 = F_{\text{bicep}} + w_m - w_{arm} - F_{\text{elbow}}$$
$$\Rightarrow F_{\text{elbow}} = F_{\text{bicep}} + w_m - w_{cm}$$

The force on the elbow joint can be quite large indeed! It is comparable to the biceps. While we typically of professional weight lifters and athletes as having VERY powerful muscles they also develop extremely strong joints. However, there are, in fact, other muscles/tendons that help reduce the load on the joints and what we've found is only a first approximation. Be that at is may joints can be a weak spot in the body.

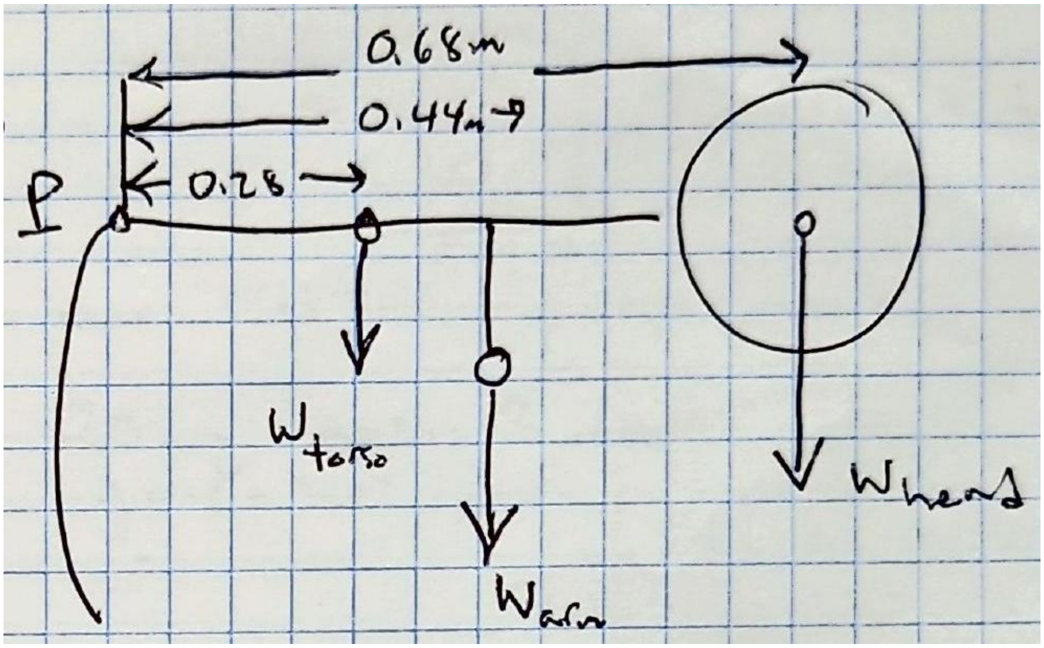
Baseball players produce HUGE forces on their joints. They amplify those forces by hitting a ball with a long stick thereby increasing the torques involved. A 162 game long season is perhaps the most grueling in all sports and players are constantly hurt.

Spine

It has haven estimated that ˜95% of all spinal injuries occur in the low back around the L4/L5 vertebral levels. Usually the protective cushion, the disc between the vertebrae, pops out or even ruptures. The consequence is severe pain and potentially paralysis. No fun.

Let us examine the forces/torques involved in the spine.

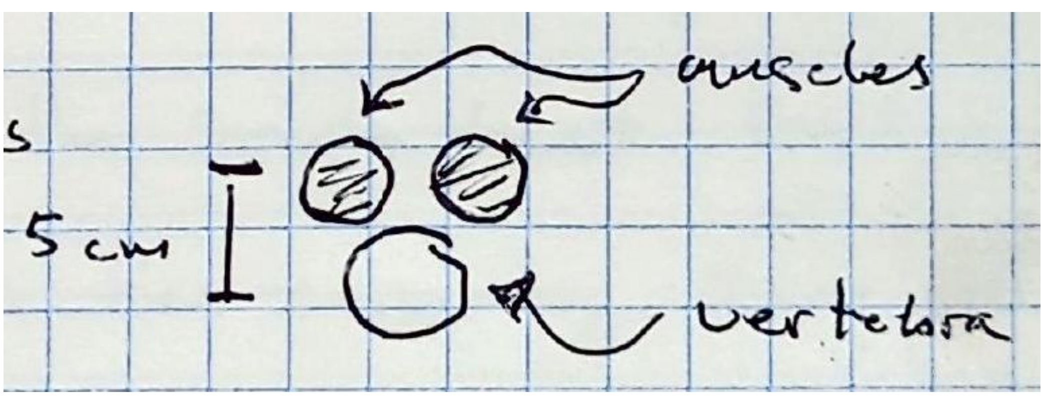
Consider a person bent over perhaps picking up a box. We estimate the various lengths and connection sites of the primary muscles involved in this situation. We choose our pivot to be near our spine. We deconstruct our weights to be the weight of the head, the arms, and the torso. see below



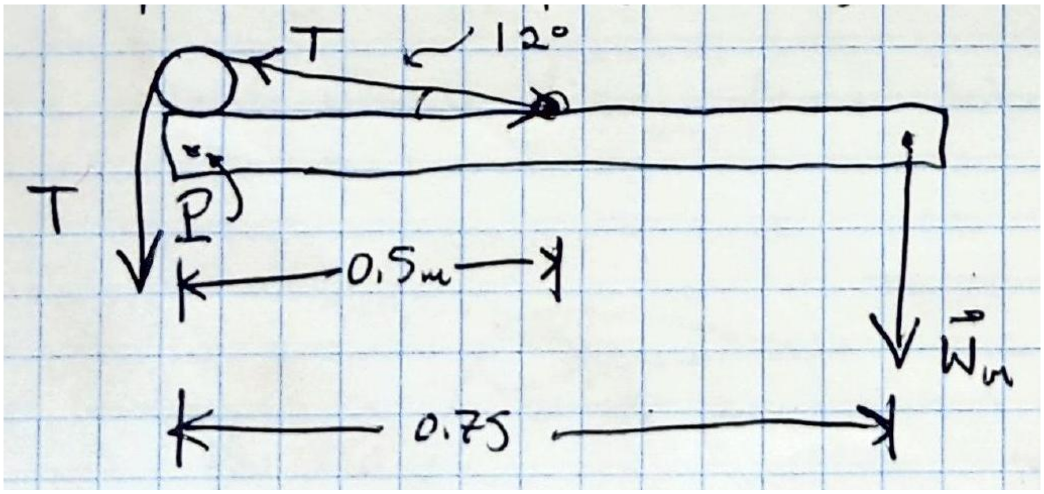
Using the diagram above along with our two conditions of equilibrium we find a torque about P of 150Nm if we lift a 45 pound box.

This really doesnt seem to bad. however, we made a MAJOR assumption. we assumed that the back muscles acted directly at the pivot. In reality this is not so. So we must redo our computation to consider the distance between the spinal column and the spinal muscles.

The large back muscles are slightly above the spinal column by about 5cm as in the figure below



Now we must consider a different geometry - see figure below

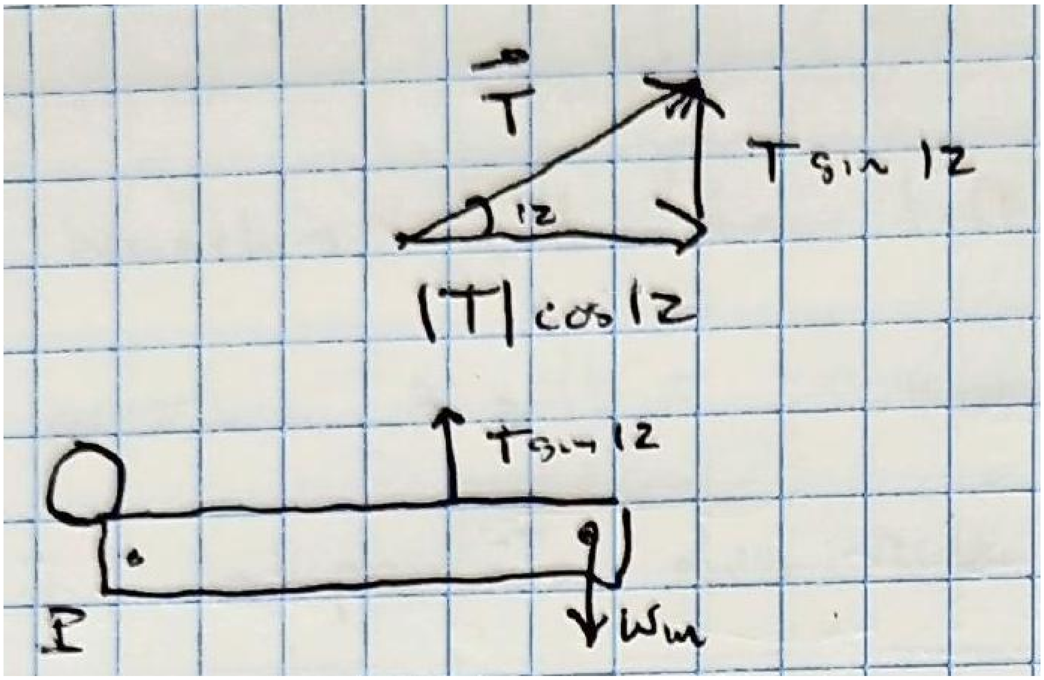


Now we must consider the tension force in the back muscles which operates at 12° at the torso. Then we must decompose everything and solve

$$\sum \tau_i = 0$$

$$\sum F_i = 0$$

To Solve we will need to decompose $\vec{T}$



the torque due to T is

$$\tau_T = 0.5T \sin 12 = \frac{T}{10}$$

The torque due to the mass is

$$\tau_m = -\frac{3}{4}m = 150$$

Plugging into our EQ condition we find

$$T = 1500$$

This is a massive amount of tension just for a 20kg mass. If it were instead 45kg then $T = 3370$ N which is roughly 700 pounds!

Physical Therapy is the single best treatment of low back injury but difficult and time consuming, Many people, with their busy lives unfortunately rely on pain medications. While these are very helpful in the short term, building muscle is the single most viable long term solution.

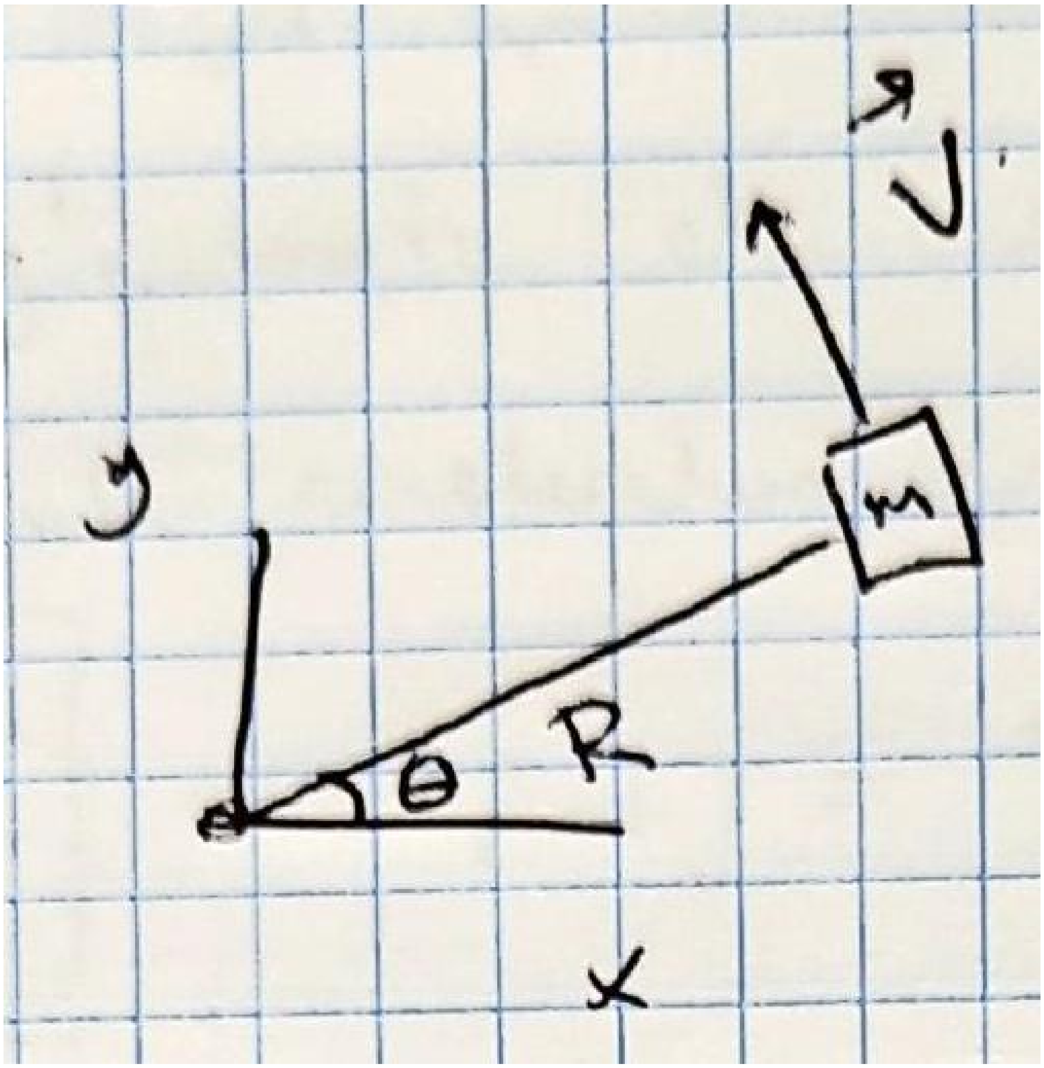
In PT doctors and medics also focus on strengthening the abdominal muscles. (the "core"). These muscles aid in lifting the load. The abdomen is under pressure and this pressure pushes back against the load. If a person's abdominal muscles are weak and they try to lift a heavy load they run the risk of that pressure pushing the intestines out through the abdominal muscles!

Dear reader, this computation may lead you to consider "always lift with your legs first and foremost"!!

Rotational Dynamics

We now arrive at one of the most challenging subjects in physics. We will stare our intuition in the face and watch as the mathematics and physical reality obliterate what we thought we knew. We will begin to rely more and more on geometric and mathematical concepts to guide our understanding of reality and after a few years of perilous confusion you'll slowly start understand this stuff.

Let us recall what we've learned about rotational physics

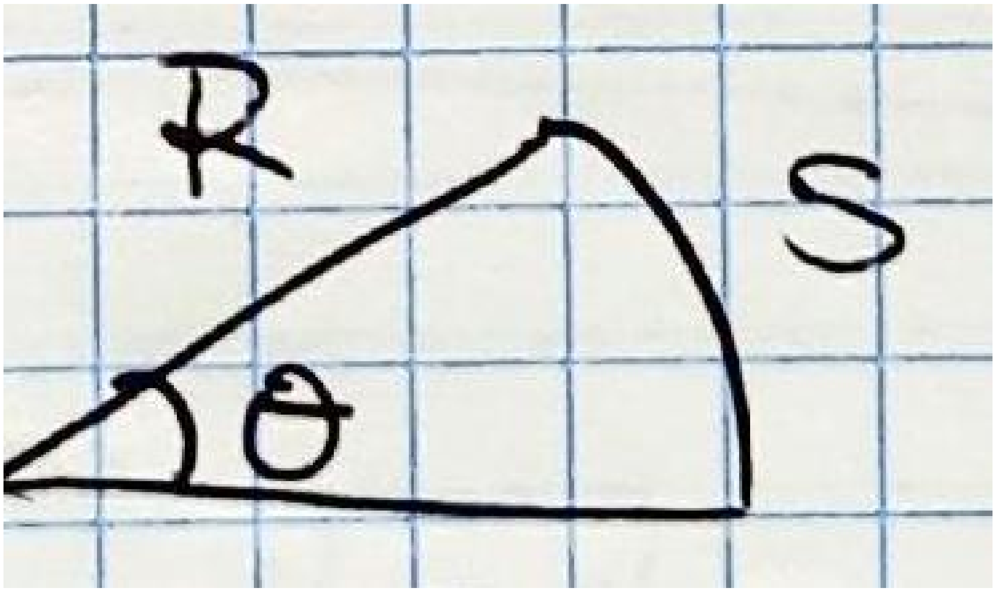


A rotating mess on a circular trajectory at constant speed experiences a centripetal force/acceleration directed directly into the center of curvature.

$$a_c = \frac{v^2}{R}$$

The change in angular position (θ) per unit time is the "angular velocity". ω

$$\omega_0 = \frac{\Delta\theta}{\Delta t}$$



The distance traveled along the arc is " s " and so

$$v = R\omega$$

If we no longer have "uniform" circular motion and the velocity/(or angular velocity) changes we say that the mars has an "angular acceleration" given by

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

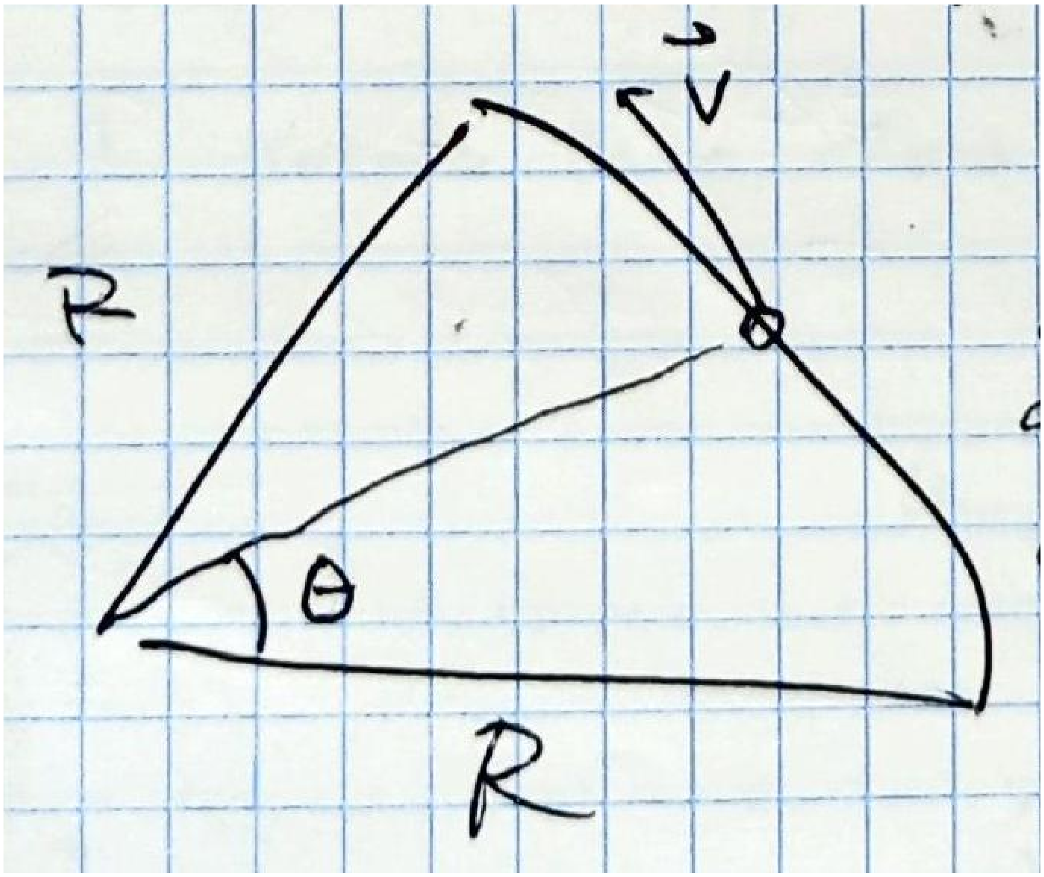
If α is constant then we can write everything in a form similar to Galileo.

$$\theta(t) = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega(t) = \omega_i + \alpha t$$

Everything up to this point has been rotational kinematics. Now we shall develop the dynamics in a manner analogous to $F = ma$. We will find torques can produce angular acceleration.

Now, on a circle with changing speed we have a new accelerations to consider.



We have an instantaneous centripetal acceleration a_c directed inward but also an acceleration due to $\frac{\Delta v}{\Delta t}$ directed tangentially to the current position

$$a_t \perp a_c$$

Now,

$$a_t = \frac{\Delta v}{\Delta t} = \frac{\Delta(R\omega)}{\Delta t} = R\frac{\Delta\omega}{\Delta t} = R\alpha$$

so that we now have

$$a_t = R\alpha \quad a_c = R\omega^2$$

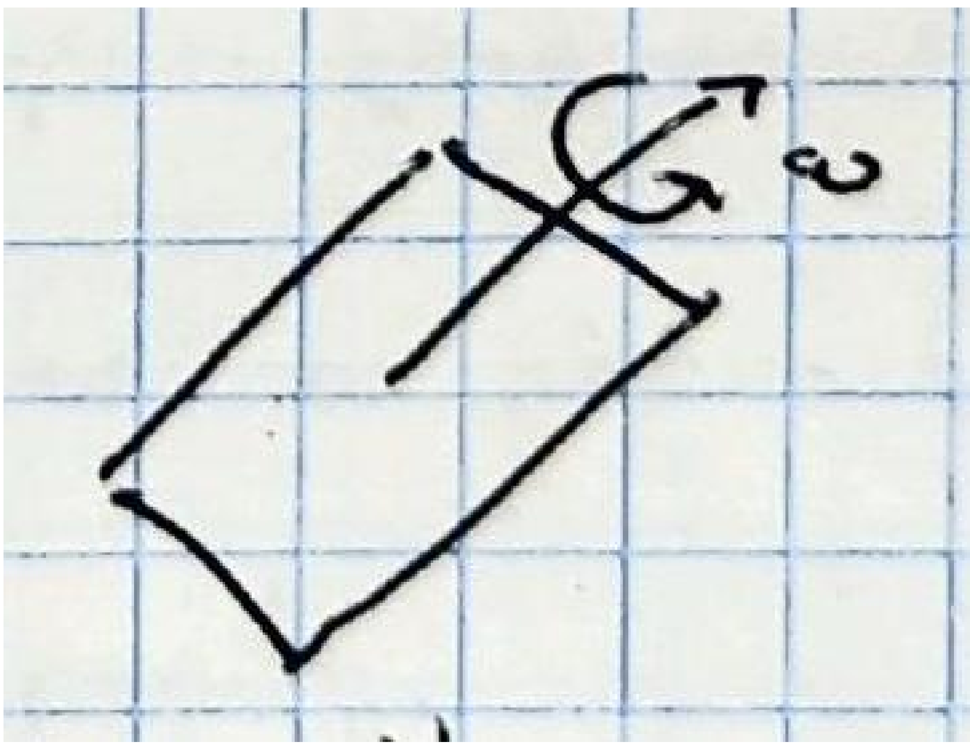
or we can say

$$\alpha = \frac{a_t}{R}$$

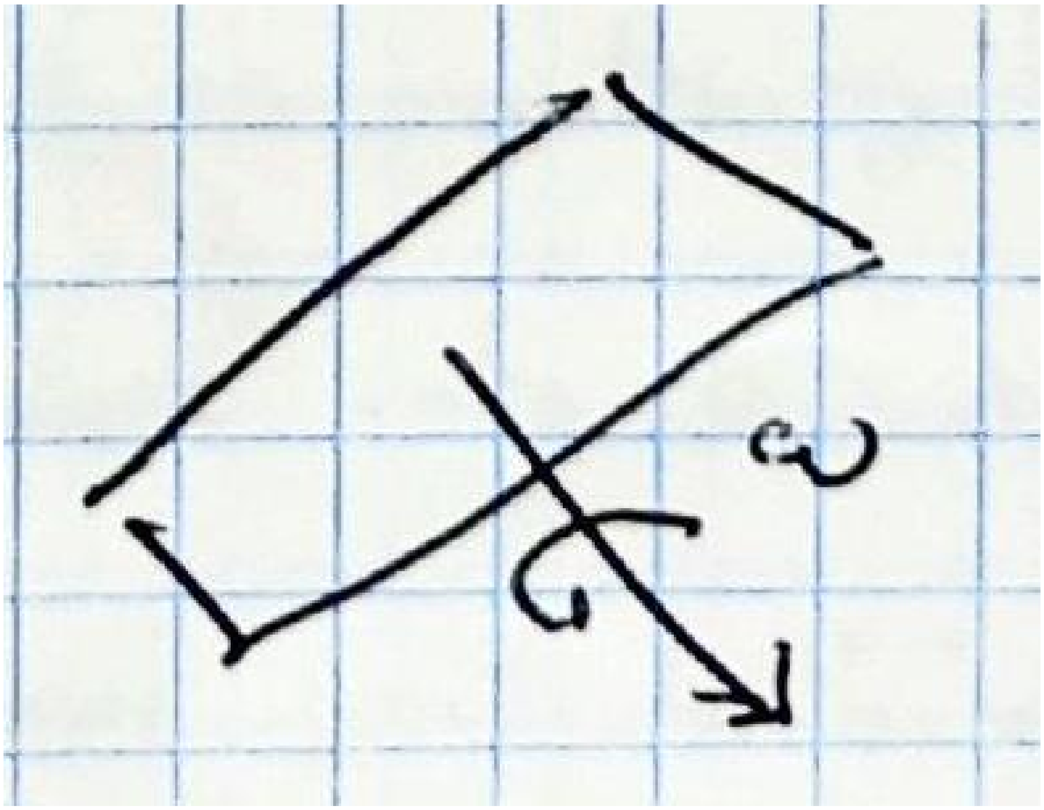
a_t is linear and due to some force acting on the object. Something is speeding it up tangentially.

Dynamics: Rotational Inertia

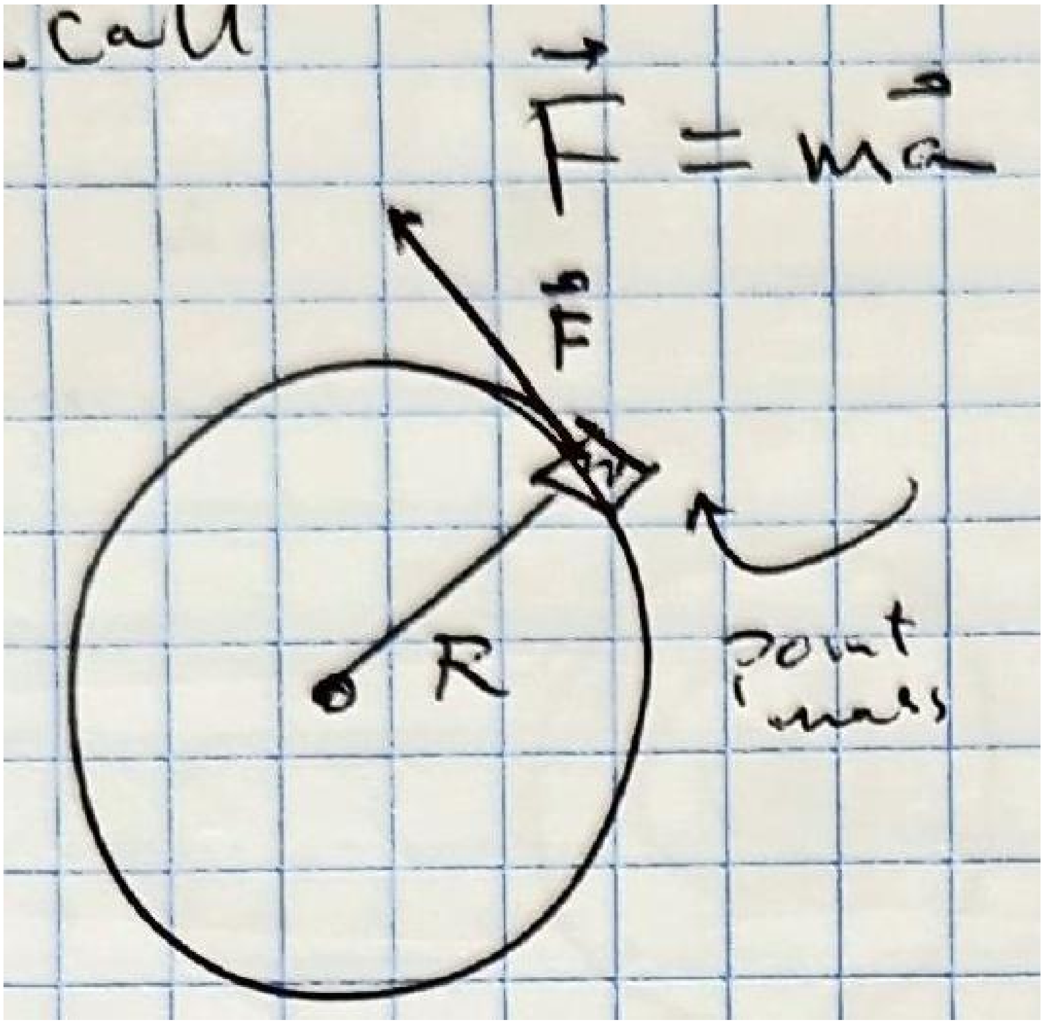
Heavy masses are hard to rotate but some SHAPES are easier to rotate than others. For example



is easier than



But why? Recall uniform circular motion:



The tangential acceleration in this case $a_t = R\alpha$ so that,

$$F = mR\alpha$$

The torque applied to the mass $\tau = RF$ if we take the pivot to be the center of the circle.

$$\tau = mR^2\alpha$$

This is analogous to $F = ma$ where m is the "inertial" mass. Similarly, here we have $\tau = I\alpha$ where $I = mR^2$ is the rotational inertia.

Heavier things are harder to spin (due to m)but also bigger things are harder to spin (due to R): I captures both aspects.

Now, a disc is nothing but a bunch of tiny individual masses all assembled into a disc so we can add all their contributions together - $\sum_i m_i R_i^2$.

In fact, the tools of calculus give us a way to find I for any particular shape. A single point mass was straight forward but the formulas for I of different shapes can be VERY complicated.

We will only be working with a few standard shapes such as spheres, balls, cylinders, and rings to name a few.

Now, in statics, or equilibrium, we said

$$\sum \vec{\tau}_i = 0$$

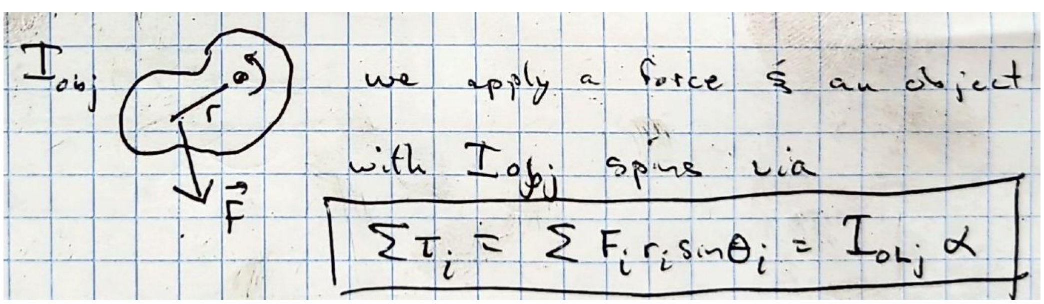
However, here we are saying, given a tangent force

$$\sum \vec{\tau}_i = I\vec{\alpha}$$

If we apply a force to a wheel it will speed up (accelerate) $\sum \tau_i = I\alpha$ tells us that it is easier to accelerate an object if most of its macs is near the axis. This is due to $I \propto MR^2$.

If, however, we take the same total mass but arrange it into a smaller radius R then I becomes much smaller and the mass is easier to rotate.

Typically we solve general problems as in the figure below:



If we know where the force is applied then we can find the torque and if we know the torque we can is solve for α and then we have everything we need for the motion of the object.

All of this is entirely analogous to what we did when solving $F = ma$ problems

$$\sum \vec{F} = m\vec{a}$$

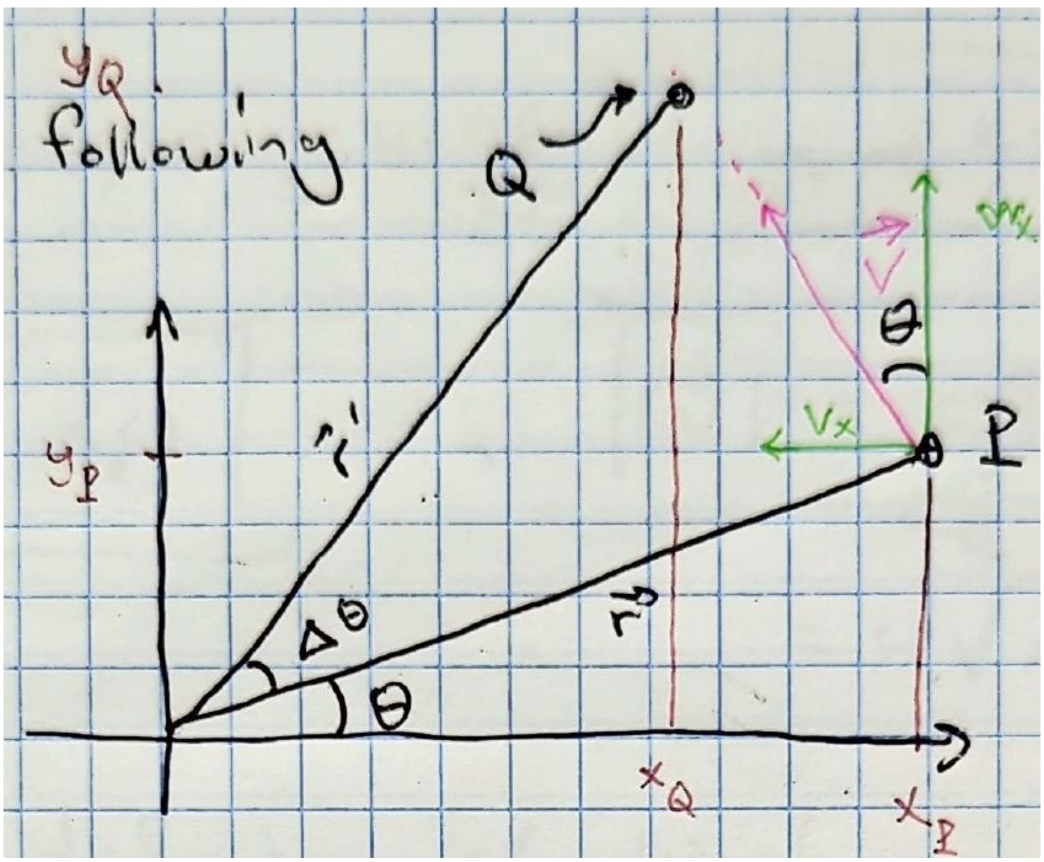
Choose coordinates and add up all forces and solve for $\vec{a}$

$$\sum \vec{\tau} = I\vec{\alpha}$$

Choose coordinates AND a pivot add up all the torques and solve for $\vec{\alpha}$

Rotational Energies

Consider the following diagram



A particle moves from P to Q.

Now from P to Q the x coordinate changes by $\Delta x = x_Q - x_P$ and y changes by $\Delta y = y_Q - y_P$.

But, we have that

$$\Delta x = -\overline{PQ}\sin\theta$$

We also have,

$$\sin\theta = \frac{y_P}{R}$$

and $\overline{PQ}$ is approximately the arclength between P and Q if we consider $\Delta\theta$ to be tiny.

The $\overline{PQ} = R\Delta\theta$. Now we can plug into our form for Δx and find

$$\Delta x = -R\Delta\theta\sin(\theta) = -R\Delta\theta\frac{y_P}{R} = -y_P\Delta\theta$$

We analyze Δy similarly so that we have

$$\Delta x = -y_P\Delta\theta$$

$$\Delta y = x_P\Delta\theta$$

Now, suppose the object is rotating with an angular frequency of ω. Then we have

$$v_x = \frac{\Delta x}{\Delta t} = -y_P\frac{\Delta\theta}{\Delta t} = -y_p\omega$$

And similarly

$$v_y = x_P\omega$$

and the magnitude of $\vec{v}$ is $|\vec{v}| = \omega R$

Now let us recall our definition of work.

$$\Delta W = \text{ Force } \cdot \Delta \text{ distance}$$

Here the distance is Δx and Δy.

Work is done in the x-direction and the y-direction and then the total work is the sum of the two.

$$\Delta W = F_x \Delta x + F_y \Delta y$$

but we found Δx and Δy so that plugging in we have

$$\Delta W = \left(x_p F_y - y_p F_x\right) \Delta\theta$$

Where, recall $\Delta\theta \ll 1$

$xF_y - yF_x$ is another way of defining torque. In fact, this combination shows up in the mathematical form of the cross product. This makes sense because $x \perp F_y$ and $y \perp F_x$

Therefore we have found the work done in rotating an object about some axis or pivot.

$$\Delta W = \tau \Delta\theta$$

Here, this torque is about the origin. We may have torques about other points if several forces are acting on an object. In that case we would add up all the torques and find

$$\Delta W = \sum_i \vec{\tau}_i \Delta\theta$$

But we've previously found that torques can generally produce angular acceleration in proportion with the rotational inertia I so that

$$\Delta W = \sum_i \vec{\tau}_i \Delta\theta = I\alpha\Delta\theta$$

This is a kind of rotational work. Rotating a mass through angle θ requires energy.

Now, let's follow the same path we took when we studied linear motion and work. There we found the celebrated work energy theorem and if nature is kind perhaps she has a rotational analog to that.

Let's look at Galileo's $\omega_f^2 = \omega_0^2 + 2\alpha\Delta\theta$ and solve for $\alpha\Delta\theta$

$$\begin{aligned}\alpha\Delta\theta = \frac{\omega_f^2 - \omega_i^2}{2} &\Rightarrow \Delta W = \frac{I}{2}\left(\omega_f^2 - \omega_i^2\right) \\ &\Rightarrow \Delta W = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2\end{aligned}$$

this looks very similar to

$$\Delta W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Let us define analogously to our KE

$$KE_{rot} = \frac{1}{2}I\omega^2$$

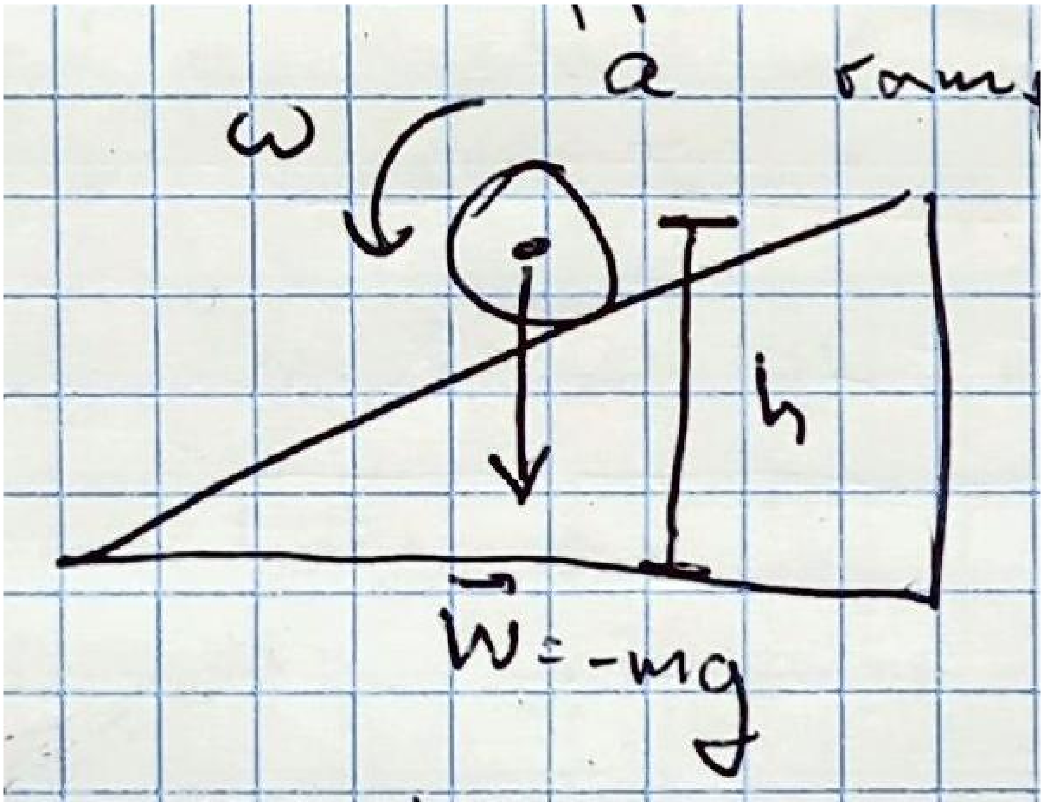
This is called the Rotational Kinetic Energy. It comes from the KE of every little mass an object is composed of. For a ball rolling then there is linear KE of the center of the ball but there is also energy contained within the rotation of the ball.

We solve our problems just as before but now we keep track of anything rotating and take note of those energies. For closed systems we use conservation of Energy/momentum

$$KE_i + PE_i = KE_f + PE_f$$

Example: Rolling Down an Incline

Suppose we roll a solid cylinder from rest down an incline as below



We are near the Earth's surface so we have $PE = mgh$. Furthermore we shall assume drag. As a subtle point we MUST have friction in order to engage in rolling motion but in this problem (and most rolling problems) we will say our cylinder "rolls without slippage".

Furthermore, for the potential energy computation we consider the height of the center of mass of the cylinder.

Now, when it gets the bottom it will have some final speed. At that point all of the PE has been converted into KE.

$$PE_i = KE_f$$

But the KE is composed of both linear and rotational energies. Therefore, COE says:

$$mgh = \frac{1}{2}I\omega_f^2 + \frac{1}{2}mv_f^2$$

Now we have the question of "how fast is it rotating" and "how fast is the linear velocity of the center of mass"?

We have one equation but two unknown quantities (we secretly know I for a cylinder of radius R and length L:

$$I_{cyl} = \frac{1}{2}mR^2$$

Interestingly it doesn't matter how long the cylinder is!

Since we cant solve this perhaps there's some rotational analog to linear momentum. Then if it turns out to be conserved then perhaps we could find another equation with which to solve this rolling problem. Back to the drawing board.

Angular Momentum

Recall

$$\sum \vec{F} = \frac{\Delta \vec{p}}{\Delta t}$$

We can also find an analogous relation for rotational motion and torques. We seek an expression like

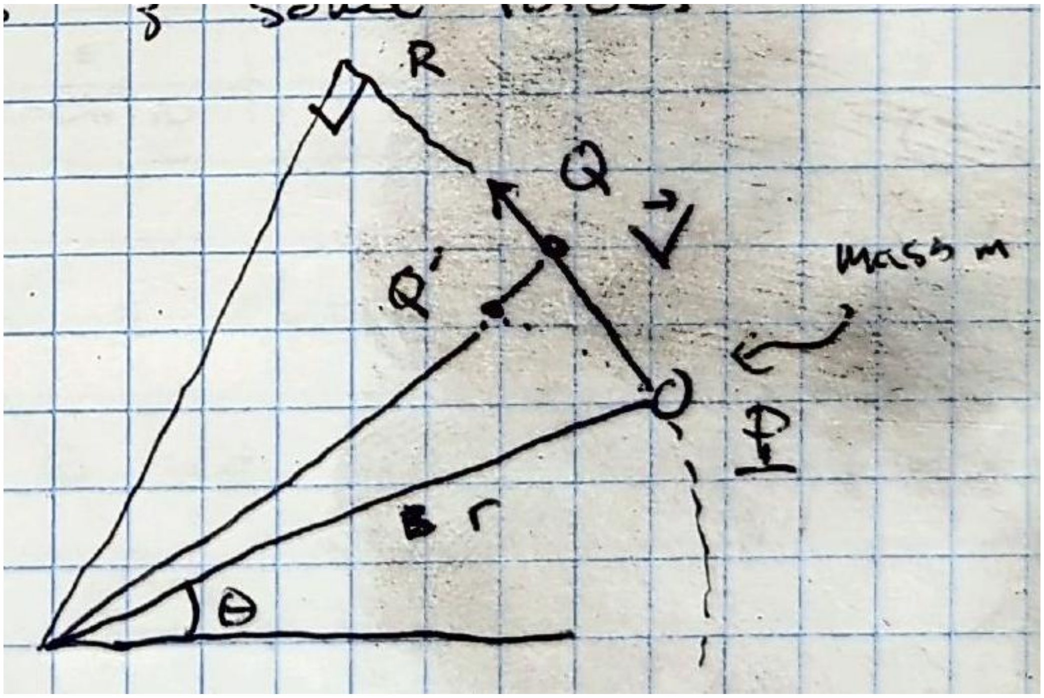
$$\sum \vec{\tau} = \frac{\Delta \vec{L}}{\Delta t}$$

Such that we hope this new thing called L is conserved like momentum is.

Let's start from geometry and proceed

We have a system of masses with some forces acting such that the object moves from point P to point Q around some axis taken to be the origin

In general we do not assume that the mass is moving circularly! This is entirely general. See below



Much of our geometry proceeds in the manner we navigated the computation of work and energy Here, because of the presence of forces we have a net torque about the origin given by

$$\tau = xF_y - yF_x$$

But we know what to do with F_x and F_y. they are ma_i so that

$$\tau = x(ma_y) - y(ma_x) = xm\frac{\Delta v_y}{\Delta t} - ym\frac{\Delta v_x}{\Delta t}$$

Notice that we now have v_x and v_y which themselves are $\frac{\Delta x}{\Delta t}$
So, in a sense we might think of

$$a = \frac{\Delta v_y}{\Delta t} = (\frac{\Delta}{\Delta t})\frac{\Delta y}{\Delta t}$$

This is admittedly funny looking however in calculus it is a standard way of writing such changes. Perhaps there is some sense with which in our formula for torque we could "factor out" a $\frac{\Delta}{\Delta t}$ and then everything else would represent our mysterious $\vec{L}$ since we would be left with an equation like $\tau = \frac{\Delta \vec{L}}{\Delta t}$

So let's try

$$\tau = \frac{\Delta}{\Delta t}\left(xm\frac{\Delta y}{\Delta t} - ym\frac{\Delta x}{\Delta t}\right)$$

but $p = mv$ sothen

$$\tau = \frac{\Delta}{\Delta t}\left(xmv_y - ymv_x\right)$$

$$= \frac{\Delta}{\Delta t}\left(xp_y - yp_x\right)$$

$$\frac{\Delta L}{\Delta t}$$

where we have $L = xp_y - yp_x$!!

We did it! Now, in general this looks like a cross product between position and momentum. Indeed when we generalize to full 3D space we have the definition

$$\vec{L} = \vec{r} \times \vec{p}$$

This is called "angular momentum". It completely depends on your choice of coordinates. An object doesn't have to be rotating to have angular momentum. For example, an object moving at some velocity at a constant y-coordinate that's non-zero, by definition will have an angular momentum "about" the origin.

However, most of our studies will be apropos rotating systems but this subtlety is important to keep in the back of your mind.

Now $\vec{p}$ and $\vec{r}$ are linear type quantities and we can always convert to polar coordinates. When we do we find another expression for $\vec{L}$

$$\vec{L} = I\vec{\omega} = I\frac{\Delta\vec{\theta}}{\Delta t}$$

This is entirely analogous to $\vec{p} = m\frac{\Delta \vec{r}}{\Delta t}$

Now, from our discoveries the units of angular momentum are

$$[L] = \frac{kgm^2}{s}$$

Theorem: For closed systems angular momentum is conserved!

This is one of the most important concepts in all of physics. It is very general. What we mean is that if $\sum \tau = 0$ then $\frac{\Delta L}{\Delta t} = 0 \Rightarrow L$ is constant

For example, if we are spinning in an office and holding a weight out stretched then initially we have angular momentum of

$$L_i = I_i \omega_i$$

remember I depend on how far away it is from the axis!

Now if we bring the mass close to our body we are then DECREASING our rotational inertia I_f where $I_f < I_i$.

But since L is conserved then in order to keep L constant then ω_f must be greater than ω_i and we spin faster!

we have

$$\Delta L = L_f - L_i = 0$$

$$\implies I_f \omega_f = I_i \omega_i$$

but

$$I_f < I_i \implies \omega_f > \omega_i$$

Angular collisions

Recall the following

$$L = I\omega$$

$$\tau = I\alpha$$

$$\omega = \frac{v}{R}$$

Then we have,

$$L = I\frac{v}{R}$$

Suppose we toss a disk with $v_i = 10$ m/s (from 5 meters away)to someone who catches it with their hand out stretched. The length of their arm is l and has a mass of $m_{arm} = 3$kg

This is clearly a perfectly inelastic collision (PIC) but we also have conservation of angular momentum (COAM), COE, and COM:

$$\sum \vec{L}_i = \sum \vec{L}_f$$

$$\sum \vec{p}_i = \sum \vec{p}_f$$

$$\sum E_i = \sum E_f$$

Consider the figure below:

Before the collision the disc has some angular momentum about the pivot which we take to be the axis of the person's torso who is catching the disc. We represent the disc as a point mass m and is a distance R from out pivot point.

Relative to our pivot P the disc has an angular speed of

$$\omega = \frac{v}{R}$$

Such that the initial angular momentum is

$$L_i = mvR$$

Now, after the collision the arm will have some angular speed and momentum given by

$$L_f = I_f\omega_f$$

Then COAM says

$$mvR = I_f\omega_f$$

But I_f is the sum of the disc and arm rotational inertia which we can look up to find

$$I_f = ml^2 + \frac{m_{arm} l^2}{3}$$

So that the rotational speed of the arm will be found by

$$\omega_f = \frac{mvR}{ml^2 + \frac{m_{arm} l^2}{3}}$$

Gyroscopes

We now must face the music: if we wish to understand the universe then we're just going to accept that we must learn to handle 3D vectors. Our aim is to understand the phenomena of gyroscopes but more importantly get a visualization of how these rotational quantities can be vectors. Previously when studying 2D systems we claimed torque is positive for CCW rotations and negative for CW rotations. I also briefly mentioned that for rotations in the x, y plane the torque vector points along the z axis. If the rotation is CCW then the torque points yp z and if CW then torque points down z. In the most general case torques can have x, y, z components in which case we need mathematical machinery to handle these complexities.

Recall our definition of torque

$$\vec{\tau} = \vec{F} \times \vec{r}$$

In the most general case we could write out our x, y, z components and perform the "cross product" vector operation and find

$$\vec{\tau} = \vec{F} \times \vec{r} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \times \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} F_y r_z - F_z r_y \\ F_z r_x - F_x r_z \\ F_x r_y - F_y r_x \end{bmatrix}$$

What a mess! This is the most general formula: if you know your Cartesian components of all your forces and positions then plugging in those values will provide you the full three dimensional torque.

Now, what we've typically been doing is considering situations defined as

$$\vec{F} = \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix}$$

$$\vec{r} = \begin{bmatrix} r_x \\ r_y \\ 0 \end{bmatrix}$$

i.e. there are no z components. Then, if we plug this into our cross product formula we find the torque is given by

$$\vec{\tau} = \begin{bmatrix} 0 \\ 0 \\ F_x r_y - F_y r_x \end{bmatrix}$$

Note, importantly, massively importantly, that the torque vector is ALWAYS completely perpendicular to both the force $\vec{F}$ and the position $\vec{r}$ in ALL cases! This will be required in our visualizations and intuitions.

Now, we also discovered

$$\sum_i \vec{\tau}_i = \frac{\Delta \vec{L}}{\Delta t}$$

Since $\vec{\tau}$ is a vector then $\vec{L}$ must also be a vector in the same direction as $\vec{\tau}$.

We previously defined $\vec{L}$ as

$$\vec{L} = \vec{r} \times \vec{p}$$

Now let us write this in its full 3 dimensional glory

$$\vec{L} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \times \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} r_y p_z - r_z p_y \\ r_z p_x - r_x p_z \\ r_x p_y - r_y p_x \end{bmatrix}$$

But we also know $\vec{L} = I\vec{\omega}$ which means $\vec{\omega}$ is also in the same direction as $\vec{L}$!

Now, for the remainder of this text we shall not need such complicated formulas - most of our vectors will be in the x, y plane and any angular momenta and torques and such will be perpendicular to those. In everything to follow it is up to you dear reader to keep track of all these directions and how they relate to each other. I will provide diagrams as best as i can and will point out any treacherous territory.

we will only consider $\vec{\sigma}\dot{\vec{s}}\vec{p}$ to be in x, y plane.

Now, let us think about what the rotational version of Newton's second law says:

$$\sum_i \vec{\tau}_i = \frac{\Delta \vec{L}}{\Delta t} = I\vec{\alpha}$$

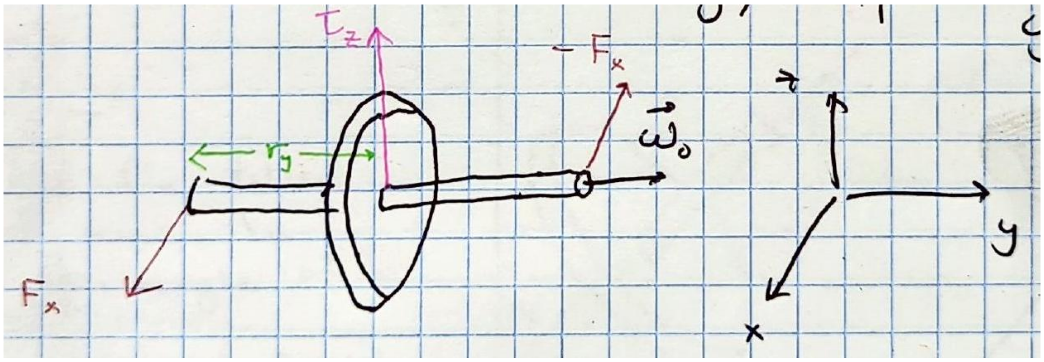
Conceptually this says "torques induce changes in angular velocity" or "torques change the angular momentum". If an object is spinning about an axis and you find its angular momentum vector either changes direction or length, then there must be a torque responsible.

Furthermore, torques are 90° to the force producing them. We will see examples where this fact produces confounding and shocking effects!

Consider a gyroscope with axis pointed along the y-direction as in the figure below

We must be able to visualize this in three dimensions. If you have a gyroscope, go get it, I'll wait.

Suppose the gyro is rotating about its axis with some angular speed ω_0. Then, from our diagram, this angular velocity points along the $+y$-axis. Further, the "spindle" of the gyro has length r from the disc, and the metal disc itself has radius R and mass m



That's all. We aren't doing anything else. We're not even near the earth so that weight plays no role - it's spinning out in deep space and will always spin because we invented zero friction bearings! What does Newton have to say?

$$\sum_i \vec{\tau}_i = \frac{\Delta \vec{L}}{\Delta t} = I\vec{\alpha} = I\frac{\Delta \vec{\omega}}{\Delta t}$$

But $\frac{\Delta \vec{\omega}}{\Delta t} = 0$ which means all the toques sum to zero - there is no net torque acting on the gyro.

Okay, that's boring. Now let's spice it up.

Suppose I grab the gyro by the spindles and push the right side of the gyro in the $-x$ direction and I pull the left side in the $+x$-direction with equal forces. We are attempting to rotate the entire gyro about the z-axis.

What, pray tell, will happen?

Well, this force we're applying, call it F_x is acting a distance r from the origin and therefore produces a torque.

let's consider each side.

Right side:

here our force is in the $-x$-direction and is trying to rotate the gyro around the z-axis in the CCW sense.

$$\tau_R = +F_x r$$

Left side:

Here we have a force in the $+x$-direction also rotating in the CCW direction about z

$$\tau_L = +F_x r$$

Then the total torque must be

$$\tau = +2F_x r$$

Because it is + it points UP the z-axis as shown in the figure.

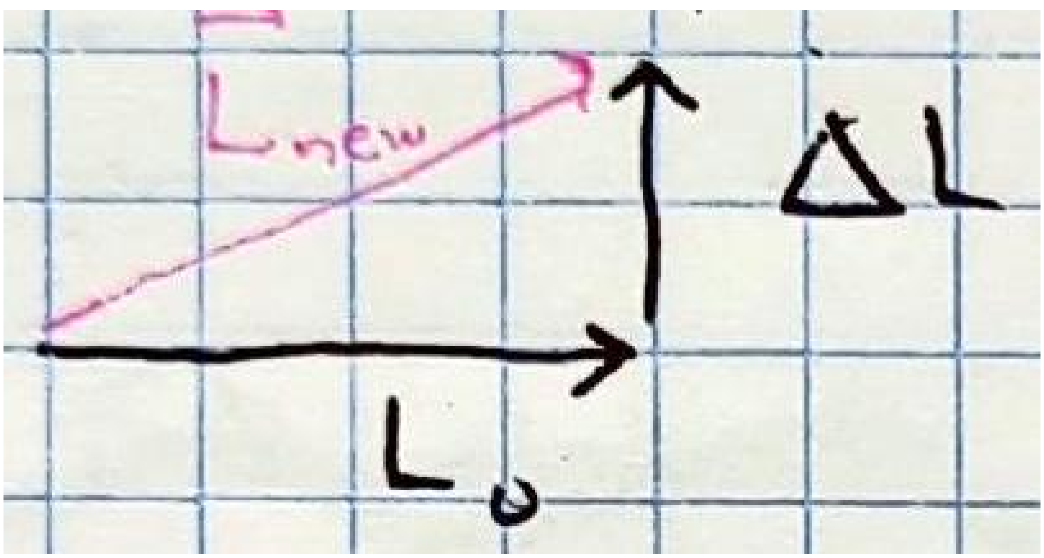
But now we have a net torque acting on our system and Newton says:

$$\sum_i \vec{\tau}_i = \frac{\Delta \vec{L}}{\Delta t}$$

So our net torque must cause the angular momentum to change. Prior to our torque $\vec{L} = I\vec{\omega}$ which we found was pointing along the $+y$-axis. Let's call its length L_0

After we're done torqueing the system for some amount of time the angular momentum will be pointing in some new direction.

Since the torque vector was pointing up z then Newton says $\Delta\vec{L}$ is also pointing up z! This means we apply a force in the x-direction and the entire system lifts up towards the z-axis!



Think about how weird this is: you push one way and it move 90° in a different direction!

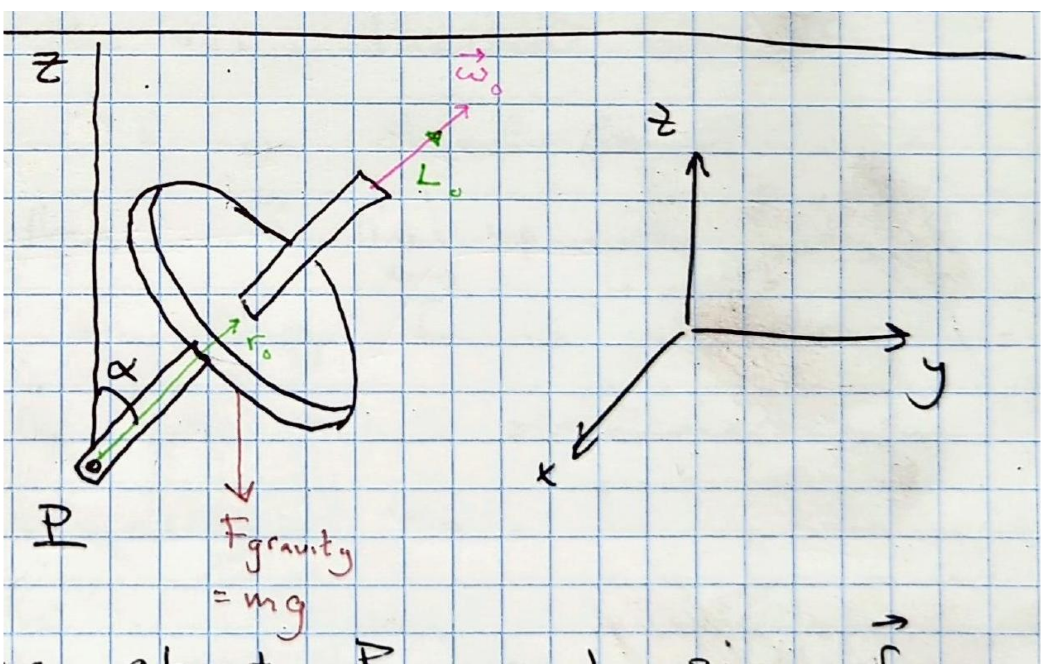
With rotational motion there's this 90° response one must always consider. A helicopter is such that you tilt the blades in the direction you'd like to go forward and the helicopter moves left or right! So engineers must build in this 90° into their controls!

This will be crucial for understanding why you can balance a gyroscope on its spindle and it will not fall. Rather it will precess around the vertical access slowly rotating its angular momentum vector around and around. Our goal is to explain why

Precession

Now we shall consider a tilted spinning gyroscope somewhere near the Earth. The angle of tilt with respect to the z-axis will be denoted α - dear reader this is NOT angular acceleration - there will be no accelerations in this study.

Now, Let us consider all forces acting on this gyro as below



Since we're near the Earth we have a force due t gravity pulling on the center of mass located at position r_0 in the figure. Therefore, gravity is applying a torque to our system in relation to the angle α.

So, since we have a net torque we expect the angular momentum vector to change. So our goal is to figure out which direction $\Delta\vec{L}$ points in.

Consider the system to be initially in (z, y) plane as in the figure

Gravity is applying a torque about P and since $\vec{r_0}$ and F_g are in the $z-y$ plane the torque will be directed along x (perpendicular to both $\vec{r_0}$ and $\vec{F_{grav}}$.

If the applied torque is along x then $\Delta\vec{L}$ will also be along x according to Newton. Therefore the gyro will move into the $-x$ direction over some Δt. But then gravity will still be applying a torque and so the gyro will rotate in the same manner at all times and will precess around the z-axis in the CCW direction since ω is CCW.

There we have it. The lesson is

1. find your net torque
2. set that to be equal to the change in angular momentum
3. figure out how the angular momentum sweeps around
4. Galileo will tell you all the kinematics if angular acceleration is constant.

Harmonic Motion

Recall Hooke's Law: Hooke said that a spring can be modeled by a linearly dependent force related to the amount stretched. e.g.

$$F = -k\Delta x$$

Where k is a constant that describes the strength of a spring.

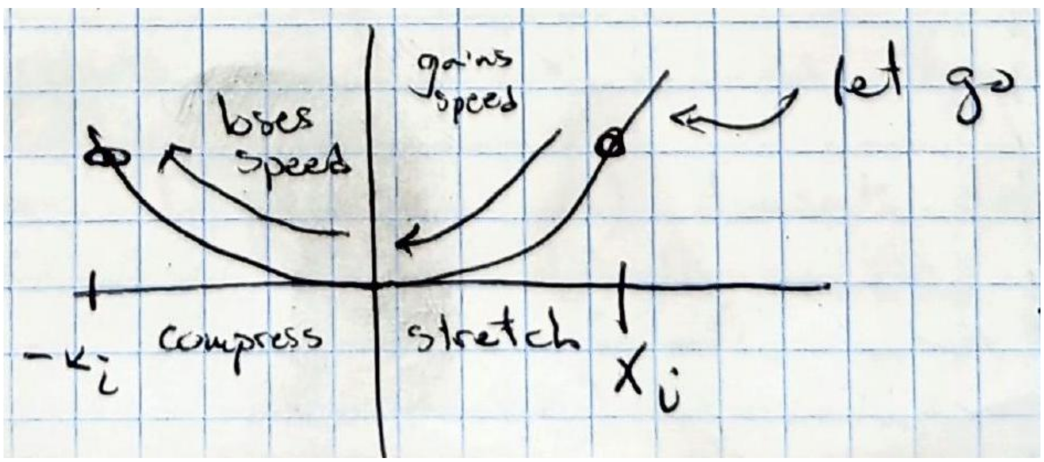
Previously, we found that the work done to stretch of compress a spring yields its potential energy as

$$PE = \frac{1}{2}k\Delta x^2$$

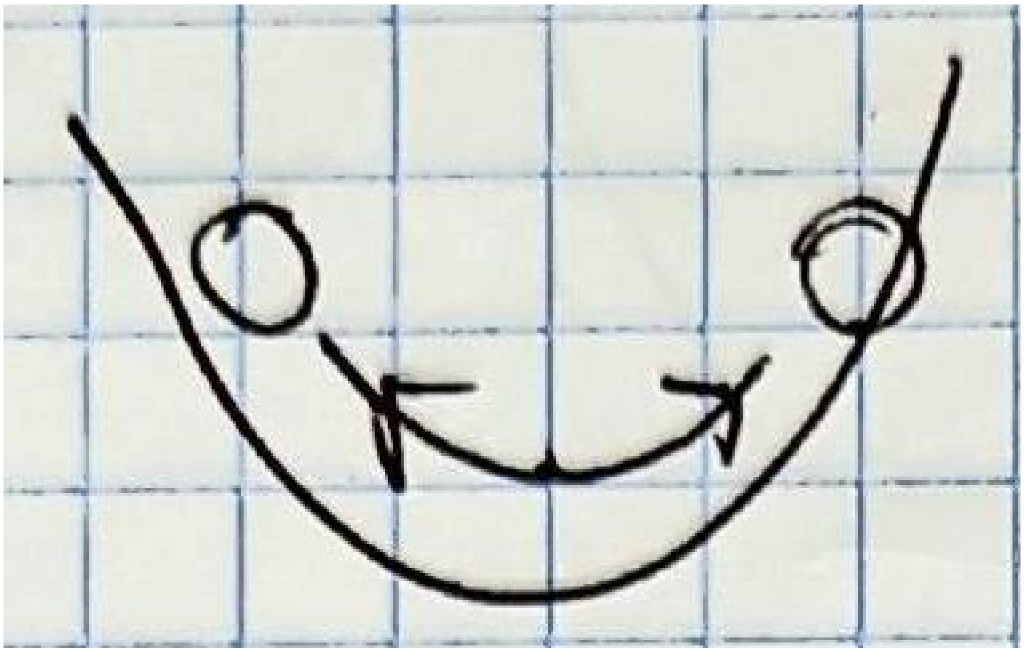
This equation is a quadratic in Δx. For ease let us write instead x to represent the stretch from equilibrium. Then $PE = \frac{1}{2}kx^2$.

If we stretch the spring and then let go our experience tells us that it will move back andd forth repeating its motion for as long as friction, drag, and dissipative forces will allow.

The potential energy is converted into kinetic energy. Then that kinetic energy compresses the spring thereby converting kinetic energy back into potential energy, and on and on the cycle repeats. See figure below



This may remind us of a ball rolling in a bowl. The ball rolls back and forth converting gravitational potential energy into kinetic energy which cycles back into gravitational potential energy.



A general feature of functions that represent potential energy, no matter what they represent or where they come from, you can imagine the potential energy function representing a bunch of hills and valleys and any predicted motion being analogous to a ball rolling along the shape like a mountain range. Balls roll down hills and systems attempt to reduce their potential energy.

Now, back to springs - it turns out that one can prove that ANY general potential energy function can be approximated by a spring potential near the bottom (the valleys) of the curve. This means if we can understand springs then we can understand all sorts of other phenomena completely unrelated to springs!

Now, we ask "how long does it take for the object to go from its starting point back to its starting point"? We will call this 1 cycle and the time it takes shall be called the period T.

The frequency of an oscillation is defined as the number of cycles per second and periods are measured in terms of seconds. The unit $\frac{1}{s}$ is called a Hertz Hz and we often describe repeating motion to be some number of Hz.

It turns at that for a spring oscillator the period and/or frequency doesn't depend on how big Δx is. Given k and m then no matter how you stretch or compress it will have a period of oscillation given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

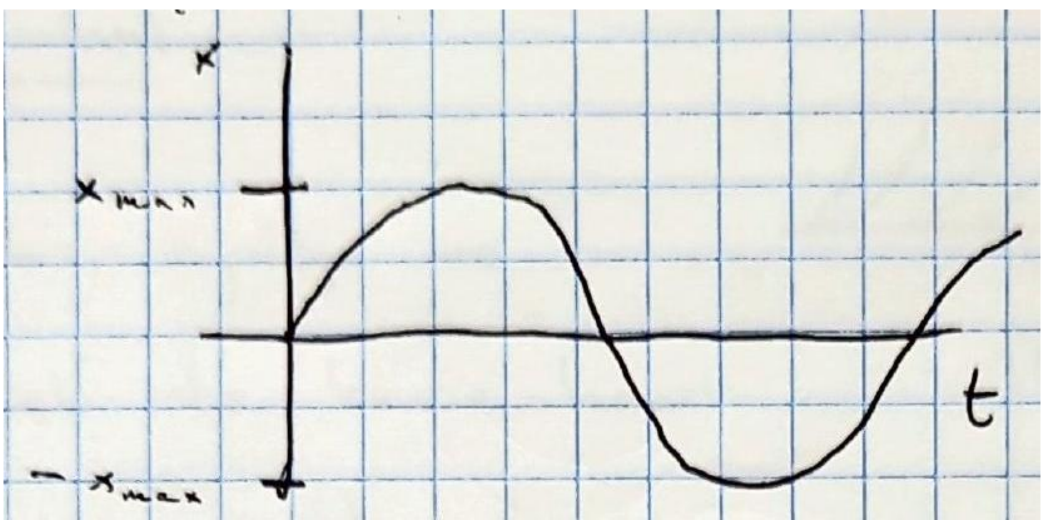
This is a fantastic result!

Now if we are interested in the (x, y)-position of the mass then we can imagine that it will oscillate between two points x_{max} , x_{min} as a function of time.

We require this position as a function of time equation to the following properties

1. doesn't decay
2. the position is the same after n cycles for $1, 2, 3 \ldots = n$
3. is bounded

a sketch of this might look like the sketch below



Clearly the simplest choice for what might represent position as a function of time would be sines and cosines. This turns out to be the case as can be shown using methods from calculus.

$$x(t) = A \cos \frac{2\pi}{T} t = A \cos \omega t$$

where ω is the angular frequency ofoscillation and is given by

$$\omega = \sqrt{\frac{k}{m}}$$

Meanwhile, the cosine function is very special because one can show that the velocity (slope of the tangent line) at any time t is

$$v(t) = -A\omega \sin \omega t$$

and similarly the acceleration is given by

$$a(t) = -A\omega^2 \cos \omega t$$

Notice, curiously that since we know the force and we know the acceleration, Newton's second law says:

$$F = -kx = ma$$

then, plugging in,

$$-kA \cos \omega t = -mA\omega^2 \cos \omega t$$

simplifying we have

$$-k = -m\omega^2$$

And a simple rearrangement of this equation gives us our definition of $\omega = \sqrt{\frac{k}{m}}$.

This cosine function is the specific function that satisfies Newton's second law. It couldn't be anything else. This is the only function with the right properties.

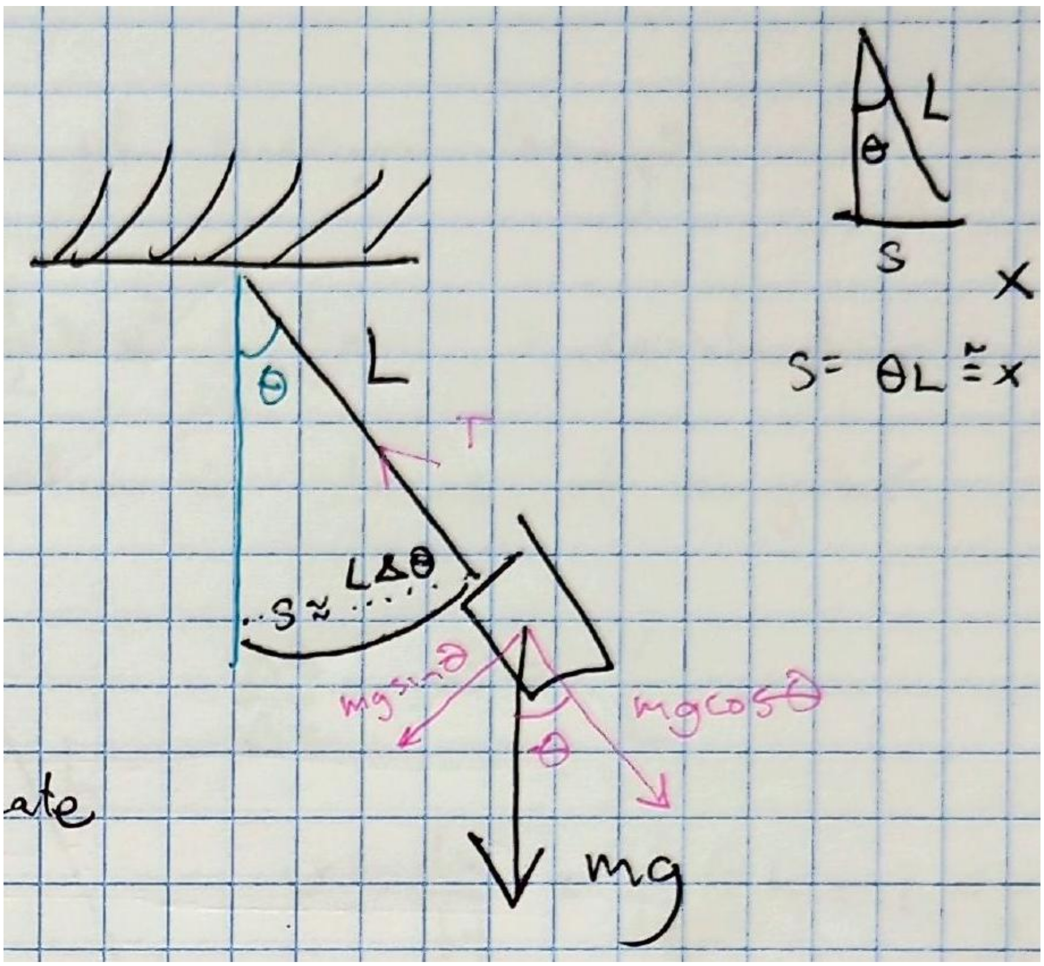
Newton's Law is a special kind of equation in calculus called a "differential equation" and they are the heart and soul of physics. Their solutions can be VERY difficult.

Almost every system in physics can be approximated by these equations if that system is near its minimum energy.

Pendulum

Another familiar system we've studied exhibits periodic motion: the pendulum. The mass swings lazily back and forth hypnotizing anyone in sight.

We can study this system effortlessly by choosing the coordinates below.



$$T = mg\cos\theta$$

$$F = ma = -mg\sin\theta$$

The force responsible for accelerating the mass is very complicated. The angle will be changing as a function of time and therefore the force is not constant and this presents a very challenging problem.

Well the spring force wasnt constant either you say and look how easily we solved that!

The conundrum is that our force is HIGHLY non-linear. for the spring we had simply $F = -kx$ but here we have a sine. Dear reader, even if I busted out my full arsenal of mathematical and physical tools I've learned over the past 25 years I would have no hope in figuring out the position as a function of time. We don't know how to write the solution to the differential equation of this system.

You're practical though and you say "but wait, I've seen a pendulum and it oscillates back and forth so what's the big deal? why cant I just try cosine or even sine?"

Ah, dear reader, you are clever indeed. Let's consider this: i want to know how this thing moves when $\theta \ll 1$ - for small small angles close to zero.

i still have

$$F = ma = -mg\sin\theta$$

but if I have tiny angles what does the sine function look like near zero? well, if you zoom in a lot eventually it becomes impossible to distinguish it from a straight line of slope 1 so maybe we can approximate the sine with $\sin\theta \approx \theta$ under the condition we don't make theta too big. what does too big mean? well its a bit fuzzy : Let's say 0.001 radians for definiteness.

Okay, i can replace my sine with just my variable θ and lo and behold we have

$$F = ma = -mg\theta$$

Dear reader, this looks exactly the same as a spring, but with different letters and you know what a spring does!

Comparing with what we know of a spring we have

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\omega = \sqrt{\frac{g}{l}}$$

where l is the length of the pendulum. This solution is a manifestation of what we meant we we claimed that ANY physical system, close to its minimum energy can be modeled as a spring!