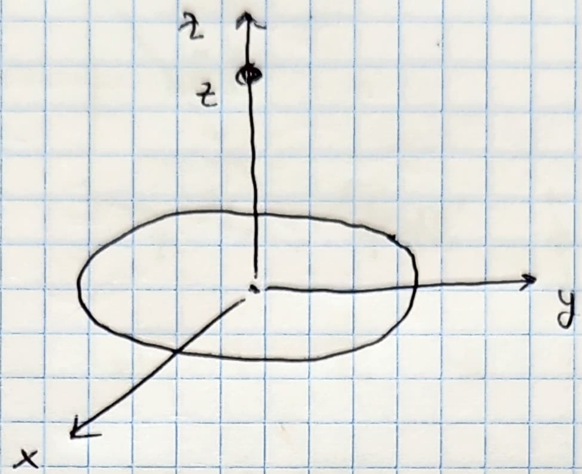
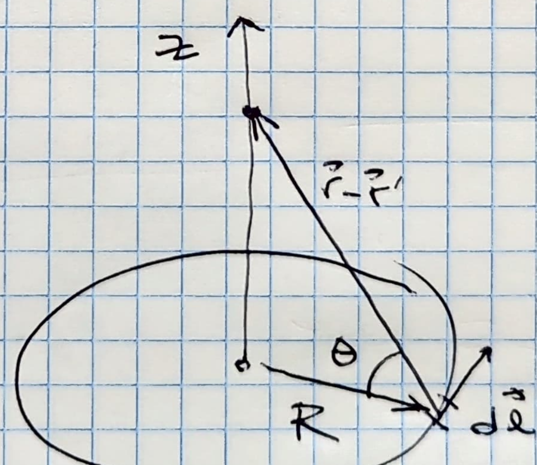
$\vec{B}$ field a distance z on axis of current loop



Biot-Savart says "add up all the elements below around the loop"

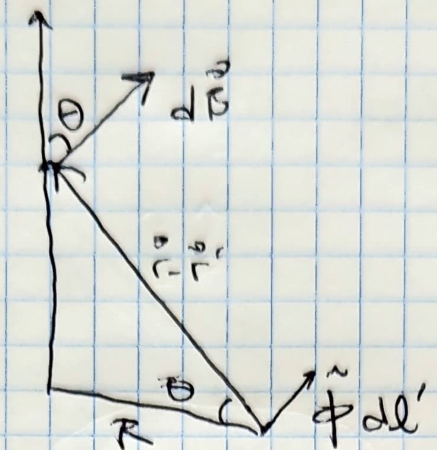
$$\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{I} \times (\vec{r}-\vec{r}\,')\, dl'}{(\vec{r}-\vec{r}\,')^2}$$

$$= \frac{\mu_0}{4\pi}\int \frac{d\vec{l}\,' \times (\vec{r}-\vec{r}\,')}{(\vec{r}-\vec{r}\,')^2}$$



now $d\vec{l}\,' = \hat{\phi}\, dl'$ and $dl' \times (\vec{r}-\vec{r}\,')$ is perpendicular to both $\hat{\phi}$ & $(\vec{r}-\vec{r}\,')$

e.g.



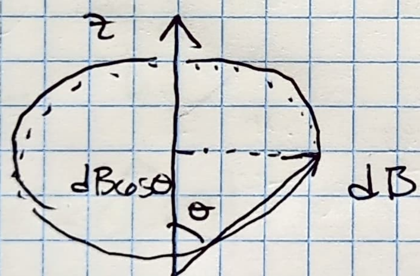
$\hat{\phi}$ is in (x,y) plane

By symmetry we expect $\vec{B}$ to be of the form

$$\vec{B} = \begin{bmatrix} 0 \\ 0 \\ B_z \end{bmatrix}$$

because as we integrate around the loop $\vec{dB}$ traces out a cone.

viz



we have then

$$B_z = \frac{\mu_0 I}{4\pi} \int \frac{dl' \cos\theta}{(\vec{r}-\vec{r}')^2}$$

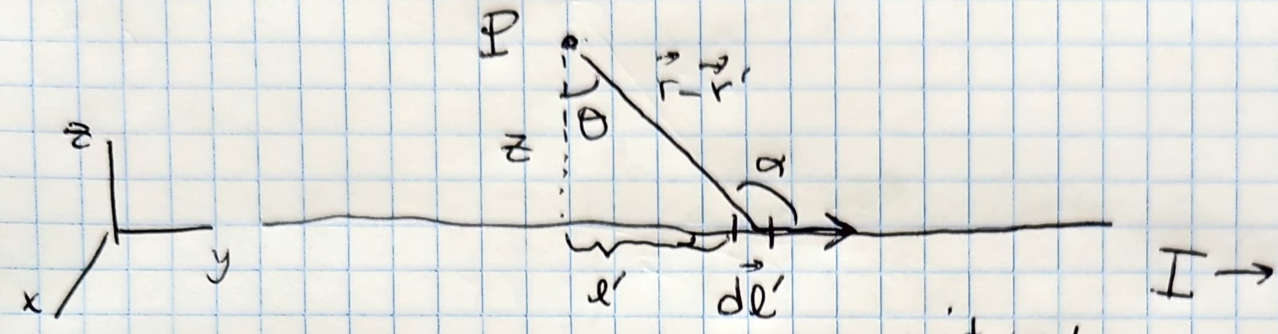
but $(\vec{r}-\vec{r}')^2 = R^2 + z^2$

and $\int dl' = 2\pi r$

rewrite $\cos\theta$

$$\therefore \quad B(z) = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{R^2+z^2} = \frac{\mu_0 I}{4\pi} \frac{R^2}{(R^2+z^2)^{3/2}}$$

B field due to a wire with Biot Savart



unit vector

① direction of $d\vec{l}' \times \widehat{(\vec{r}-\vec{r}')}$

$d\vec{l}' = \hat{y}\, dl'$ and $(\vec{r}-\vec{r}')$ is in (z,y) plane

$\therefore$ $d\vec{l}' \times \widehat{(\vec{r}-\vec{r}')}$ points out of the page in the $+\hat{x}$ direction

② the magnitude of $d\vec{l}' \times \widehat{(\vec{r}-\vec{r}')}$ is given by

$$\left| d\vec{l}' \times \widehat{(\vec{r}-\vec{r}')} \right| = |d\vec{l}'| \left|\widehat{(\vec{r}-\vec{r}')}\right| \sin\alpha$$

$$= dl' \cancel{\sqrt{z^2+l'^2}} \sin\alpha = dl' \sin\alpha$$

but here $\sin\alpha = \cos\alpha$

$\therefore$ we have

$$\boxed{dl' \sqrt{z^2+l'^2} \cos\theta}$$

we also see that $l' = z\tan\theta$

$$\Rightarrow dl' = \frac{z}{\cos^2\theta} d\theta$$

but also $(\vec{r}-\vec{r}\,')^2 = \dfrac{z^2}{\cos^2\theta}$

all together

$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2} \frac{\cos^2\theta}{z^2}\,\frac{z}{\cos^2\theta}\,\cos\theta\, d\theta$$

$$\boxed{= \frac{\mu_0 I}{4\pi z}\left(\sin\theta_2 - \sin\theta_1\right)}$$