Linear circuits are exceedingly deceptively complex & it is miraculous that they can be analyzed by straightforward application of networked components such as resistors, batteries, inductors, & capacitors.

Everything you will learn & apply to linear circuits are more accurately described by Maxwell's Laws. In particular the dictum

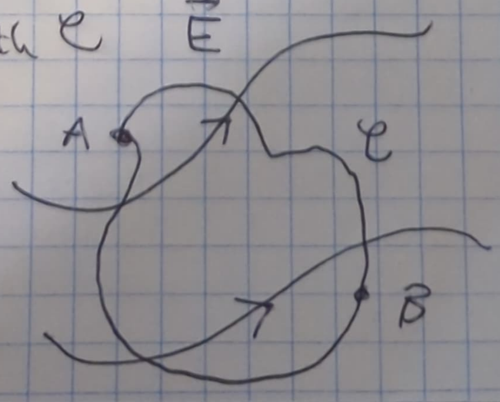
☆ "changing E-fields produce B-fields and changing B-fields produce E-fields" ☆

In this note we will focus primarily on Faraday's Law in integral form.

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi}{dt}$$

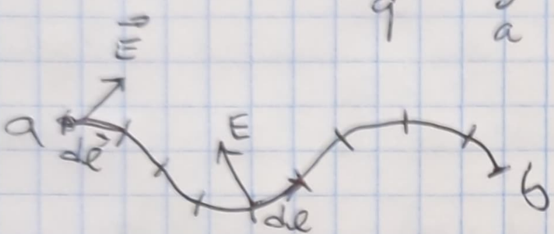
we will not ask "How is/was E created?" ~~[illegible]~~ we will only analyze its consequence

So, suppose we have some arbitrary E-field and a chosen path $\mathcal{C}$



along path $\mathcal{C}$ the work per charge done by E-field is given by

$$\frac{W}{q} = \int_a^b \vec{E} \cdot d\vec{l} \qquad \text{since } q\vec{E} = \vec{F}$$



$\int_a^b \vec{E} \cdot d\vec{l}$ tells us to

break the path up into little sections of width dl and compute its dot product with $\vec{E}$ at that point. Then we add up all such elements

If we traverse a closed path we write

$$\oint \vec{E} \cdot d\vec{l} = \frac{W}{q}$$

if $\vec{E}$ DOES NOT CHANGE in time we call

$$\int_a^b \vec{E} \cdot d\vec{l} = V(b) - V(a) \quad \text{the potential difference}$$

this is only possible because of a vector identity along with maxwells laws for static-fields

i.e $\quad \nabla \times \vec{E} = 0 \Rightarrow \vec{E} = \nabla V$

but if $\nabla \times \vec{E} \neq 0$ then $\vec{E}$ cant be written this way

Nevermind the details, the function $V(r)$ exists when $\vec{E}$ is static.

Further more!

if $\vec{E}$ is static then

$$\boxed{\oint \vec{E} \cdot d\vec{l} = 0}$$

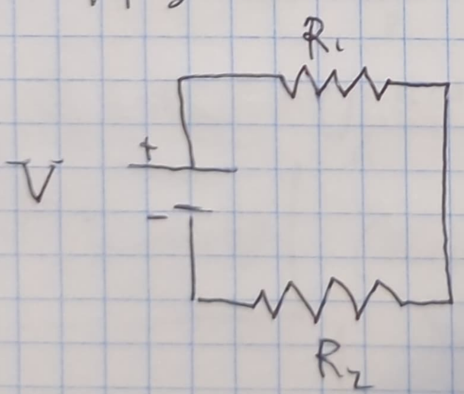
why? because we start & stop at the same point and since $\vec{E}$ is static it can be written as $-\frac{dV}{dx}$ & therefore

$$\oint \vec{E} \cdot d\vec{l} = V(a) - V(a) = 0$$

via the fundamental theorem of calculus,

$\boxed{\oint \vec{E} \cdot d\vec{l} = 0 \text{ is called "Kirchoff's Law"}}$

Let's apply it to the following linear circuit:



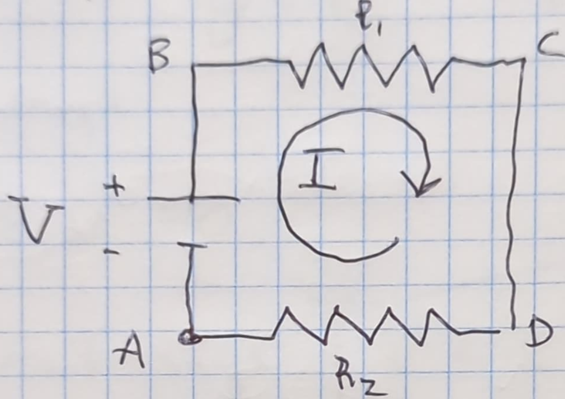
the battery produces a potential difference across its terminals. This causes a "current" to form throught the circuit.

But wait! current isn't a static E-field! All is not lost. the electrons in Cu wire at room temperature drift, on avg. at 1 cm/s so for ~~long~~ short time scales this is **ALMOST** static

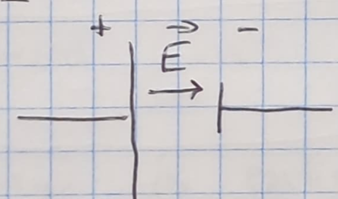
Let us analyze the circuit using $\oint E \cdot dl = 0$

and Ohm's Law $V = IR$

For our path $\mathcal{C}$ take the circuit it self



in the battery there will be an E-field pointing from + to −



Start at point A

$$\oint_{\mathcal{C}} E \cdot dl = \int_A^B E \cdot dl + \int_B^C E \cdot dl + \int_C^D E \cdot dl + \int_D^A E \cdot dl$$

$$\boxed{\int_A^B \vec{E} \cdot d\vec{l} = -V}$$ (minus b/c current/path is opposite to E-field here)

$$\boxed{\int_B^C E \cdot dl = IR_1}$$ (this is the voltage drop from Ohm's Law)

$$\int_C^D E \cdot dl = 0!$$ No E-field in a perfect conductor!

$$\int_D^A E \cdot dl = IR_2 \quad \text{Ohm's Law}$$

adding up $\oint E \cdot dl = -V + IR_1 + IR_2$

Kirchoff says

$$\oint E \cdot dl = 0$$

$\therefore$ our circuit says $\boxed{V = I(R_1 + R_2)}$

$\oint \vec{E} \cdot d\vec{l} = 0$ works for ANY circuit so long as there doesnt exist a changing magnetic flux.

Faraday's Law is <u>entirely</u> general

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

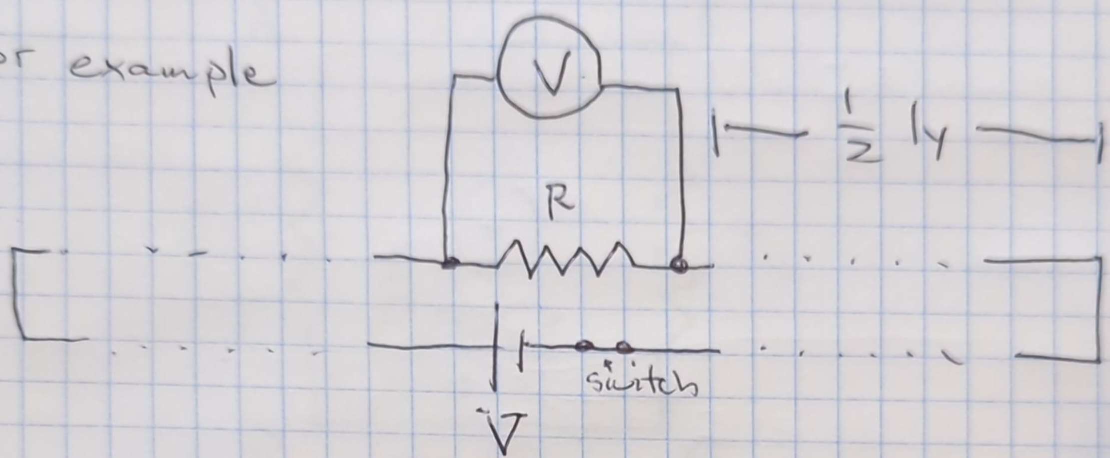
where $\Phi = \int \vec{B} \cdot d\vec{A}$

if Φ is constant then we have $\oint \vec{E} \cdot d\vec{l} = 0$ and happily use Kirchoff.

Notice that we solved this circuit by choosing a path in space EVEN THOUGH $\vec{E}$ & $\vec{B}$ field exist EVERYWHERE

This entire use of Kirchoff is APPROXIMATE the actual operation involves $\vec{E} - \vec{B}$ field traveling & bouncing around through space.

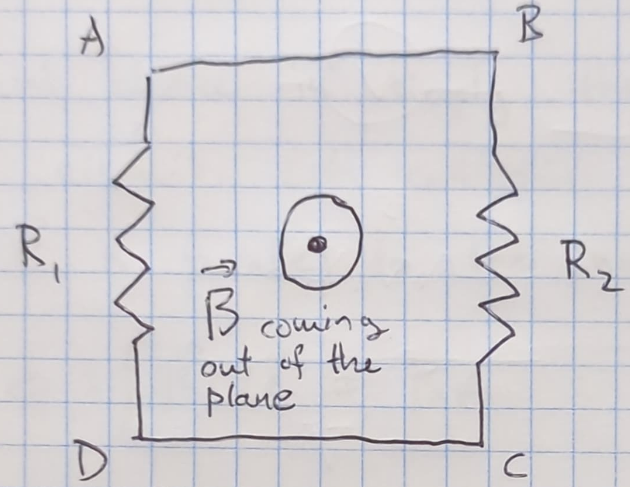
For example



this is a circuit 1 light year in length but only 1 m in width, If this circuit worked like a water pump it would take FOREVER for the little electrons moving at 1 cm/s to reach the resistor! It can, however, be shown that the time it takes to measure a voltage at R is $t = \frac{1\,m}{c\,m/s} = \frac{1}{c}\,s$!

why? because E-B fields travel through space! the metal wires act more like a reflecting box than they do water pipes though the effect of slowly moving charge is important. More on this giant circuit in a moment

But First, Let's put Kirchoff to rest by considering the following Battery-less circuit



Here we have two resistors and a <u>CHANGING</u> magnetic field pointing out of the plane towards you, dear reader.

Let's ask Faraday what he thinks,

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi}{dt} = \mathcal{E}$$

($\mathcal{E}$ is called electro motive force)

1. $\mathcal{E}$ is <u>NOT</u> a force. it has units of Volts J/C
2. $\mathcal{E}$ is <u>NOT</u> a potential or pot difference since we don't have <u>static</u> fields.
3. $\mathcal{E}$ is <u>essentially</u> a potential

Confusion abound!

Now, Φ is the magnetic flux through an open surface of our choice.

a Natural choice is simply the area bounded by the circuit. Let's call this A

since $\vec{B}$ is parallel to the normal vector of A

then $$\Phi = \int \vec{B} \cdot d\vec{A} = BA$$

if the area of the circuit doesnt change in time

then $$\frac{d\Phi}{dt} = A\frac{dB}{dt}$$

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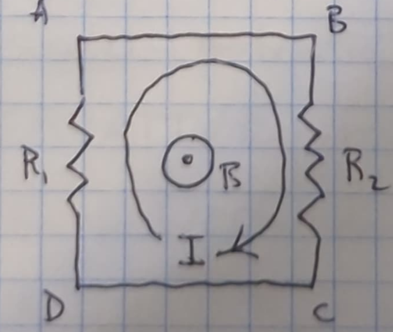
okay, let's ask "What is the potential difference between B & C points and what is the pot diffence between pts A & D?"



Now, if $\vec{B}$ is increasing out of the page then Lenz' Law says a current will induce in the clockwise direction.

$$\oint \vec{E} \cdot d\vec{l} = \int_A^B E \cdot dl + \int_B^C E \cdot dl + \int_C^D E \cdot dl + \int_D^A E \cdot dl$$

$$\int_A^B E \cdot dl = 0$$ b/c No E-field In the conductor

$$\int_B^C E \cdot dl = I R_2$$ via ohm's Law

$$\int_C^D E \cdot dl = 0$$

$$\int_D^A E \cdot dl = -IR_1$$ the minus sign is important!

☆ $V_A - V_D$ is negative b/c ☆

A is a lower potential than D

Adding them all up yields Faraday

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\phi}{dt} = \mathcal{E}$$

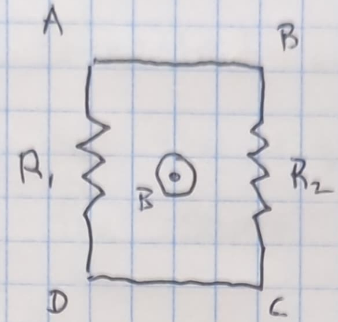
when we march around the loop we dont end up at the same potential. These are

☆ NON-CONSERVATIVE FIELDS ☆

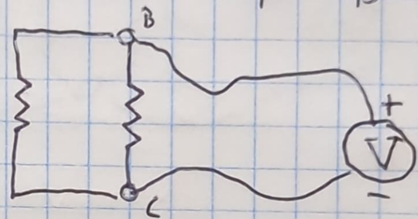
there fore, the work done is PATH-DEPENDENT

E.g. take $R_1 = 10\,\Omega$, $R_2 = 100\,\Omega$ and at an INSTANT time $t = t^*$ $I = 10\,mA$

(at a previous or later time I will be different b/c $\vec{B}$ is increasing)

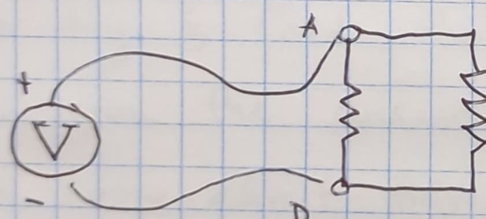


we attatch our Volt meter b/w B & C



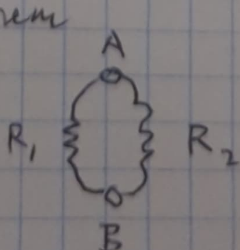
$= IR_2 = +1V$

& b/w A & D (Keeping the terminals oriented the same)

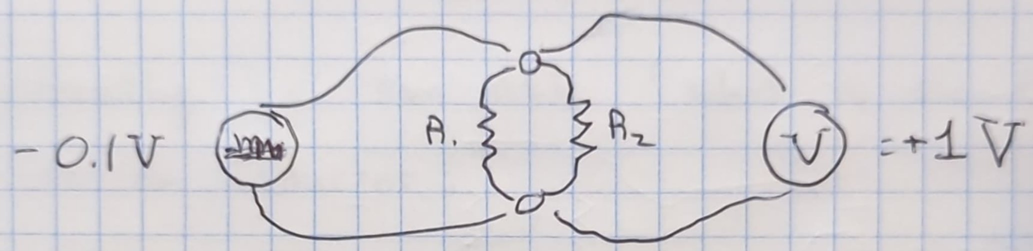


$V = -0.1\,V$ (because the volt meter opposes the current)

But the wires A−B & C−D are superconducting! It's like theyre not really there! we may as well have had the two R's connected with out them

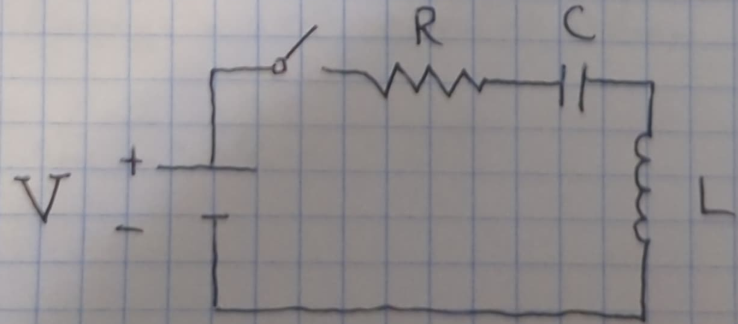


So the V-meter connected on the left measures different on the right even though they connect at the same points!



is this a paradox? Dwell upon this and try & find the resolution. I will give you the answer in time. Most folks don't know about this curiosity but experiments agree, there will be two different voltages!

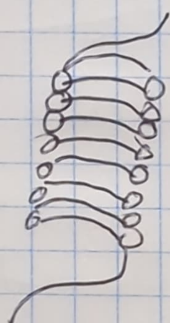
Another circuit - another subtle misconception with Kirchoff



This is the classic and important RLC circuit

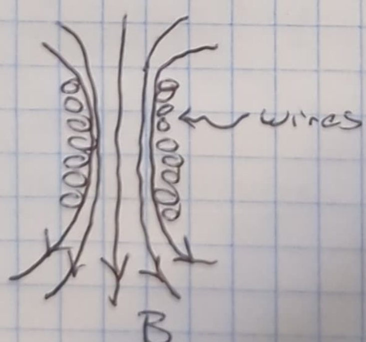
L is called an "inductor" it is just a coil of wire and "resists" <u>changing</u> E-fields.

Importantly we can ask "what is the E-field inside an inductor?"



this is a slice through a coil of wire
using gauss' Law one can prove $E = 0$ inside
It's a Faraday cage.

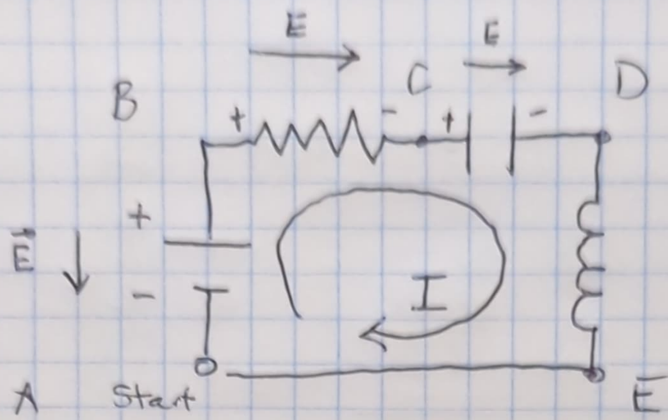
The B-field of a coil looks like



Terrible drawing!
look up a good picture of a Solenoid/coil.

<u>OKAY</u>: Analysis of circuit when the Switch is hit

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi}{dt}$$



$$\int_A^B E \cdot dl = -V$$

$$\int_B^C E \cdot dl = +IR$$

$$\int_C^D E \cdot dl = \frac{Q}{C}$$ b/c $C = Q/V_c$ & $\int E \cdot dl$ across capacitor is V_c

$$\int_D^E E \cdot dl = 0$$ Gauss' Law! no E-field

$$\int_E^A E \cdot dl = 0$$ just a wire.

add them up

$$-V + IR + \frac{Q}{C} = -\frac{d\Phi}{dt}$$

Now, a fact I wont derive

the only changing B field around is the coil/inductor it turns out that

$$\frac{d\tilde{\Phi}}{dt} = L\frac{dI}{dt}$$

L is called "inductance"

so $$\boxed{\oint E \cdot dl = -L\frac{dI}{dt} = -V + IR + \frac{Q}{C}}$$

This is TRUE!

$-L\frac{dI}{dt}$ is on one side

and everything else is on the other.

They are CONCEPTUALLY DIFFERENT

for example, we don't write Newton as

$$\vec{F} - m\vec{a} = 0$$

because $\vec{F}$ is a different physical CONCEPT than $m\vec{a}$. the math is true $F - ma = 0$ but the physics would be wrong!

unfortunately in your studies you will probably analyze this circuit using a butchered Faraday they'll call Kirchoff.

the (wrong but equivalent) Kirchoff method would find

$$-V + IR + \frac{Q}{C} + L\frac{dI}{dt} = 0$$

it "looks like" Kirchoff but it's NOT

Kirchoff is $\oint \vec{E}\cdot d\vec{\ell} = 0$

but $E = 0$ inside the inductor L

Faraday says $\oint \vec{E}\cdot d\vec{\ell} = -L\frac{dI}{dt}$

just like Newton say $\Sigma\vec{F} = m\vec{a}$

Now, for solving circuits & getting correct answers they both work fine but in complicated situations and for careful consideration of PHYSICS the Faraday way is the ONLY way. mis-applying Faraday & calling it Kirchoff where Kirchoff doesn't apply is foolish!