Critically damped harmonic motion.

suppose

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \lambda^2 x = 0 \qquad (1)$$

(the factor ~~&~~ 2 is for convenience)

this is a linear differential equation

if $x_1(t)$ & $x_2(t)$ are solutions then

$$X(t) = A\,x_1(t) + B\,x_2(t)$$

is a solution where A, B may be complex in general.

The characteristic equation is $r^2 + 2br + \lambda^2 = 0$

$$\Rightarrow \quad r = -b \pm \sqrt{b^2 - \lambda^2}$$

or we can guess an "ansatz"

$$\tilde{x}(t) = e^{i\omega t}$$

(we choose this because we expect oscillatory solutions)

Plug this guess into (1)

$$\frac{d\tilde{x}(t)}{dt} = i\omega e^{i\omega t} \qquad \frac{d^2\tilde{x}}{dt^2} = (i\omega)^2 e^{i\omega t}$$
$$= -\omega^2 e^{i\omega t}$$

putting (1) together we have,

$$-\omega^2 e^{i\omega t} + 2b\, i\omega e^{i\omega t} + \lambda^2 e^{i\omega t} = 0$$

$$= \left[\lambda^2 - \omega^2 + 2b\omega i\right] e^{i\omega t} = 0$$

$$\Rightarrow \left[\lambda^2 - \omega^2 + 2b\omega i\right] = 0$$

this is a quadratic for ω

$$\boxed{\omega = ib \mp \sqrt{\lambda^2 - b^2}}$$

we have <u>two</u> solutions for our ODE since $Dx = 0$ is linear the general solution will be a linear combination of of these solutions

$$\boxed{x(t) = Ae^{i\omega_1 t} + Be^{i\omega_2 t}}$$
$$= x_1(t) + x_2(t)$$

why? b/c $Dx(t) = Dx_1(t) + Dx_2(t) = 0$

b/c $Dx_{1,2} = 0$

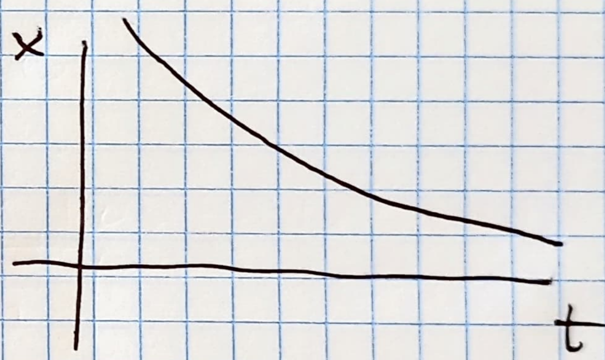
critical damping occurs when $b = \lambda$ [CD]

$$\Rightarrow \boxed{\omega = ib + \sqrt{0} = ib}$$

$\therefore$ the solution for CD is

$$\boxed{x(t) = Ae^{i\omega t} = Ae^{i(ib)t} = Ae^{-bt}}$$

a decaying exponential



check by plugging into ODE 1

$$\dot{x} = -Ab e^{-bt}$$
$$\ddot{x} = Ab^2 e^{-bt}$$

$$\dot{x} + 2b\dot{x} + b^2 x = +Ab^2 e^{-bt} + -2b^2 e^{-bt} + b^2 e^{-bt}$$
$$= 0$$
$$= b^2 - 2b^2 + b^2 = 0 \checkmark$$

determine A from initial conditions