double/triple integrals

2D: double integrals

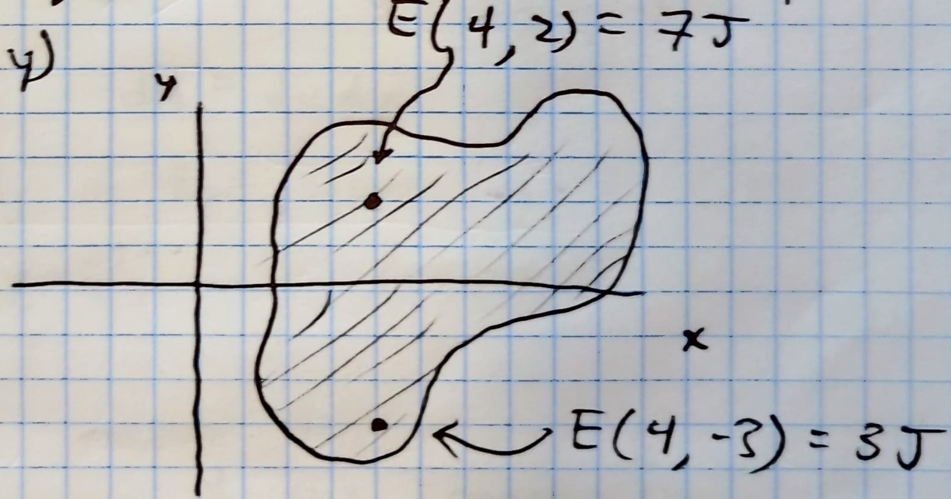
Suppose we have some region in x,y-land that we wish to integrate over

for example, we may wish to find the total energy of a region of space where the energy density is defined at every point (x,y)

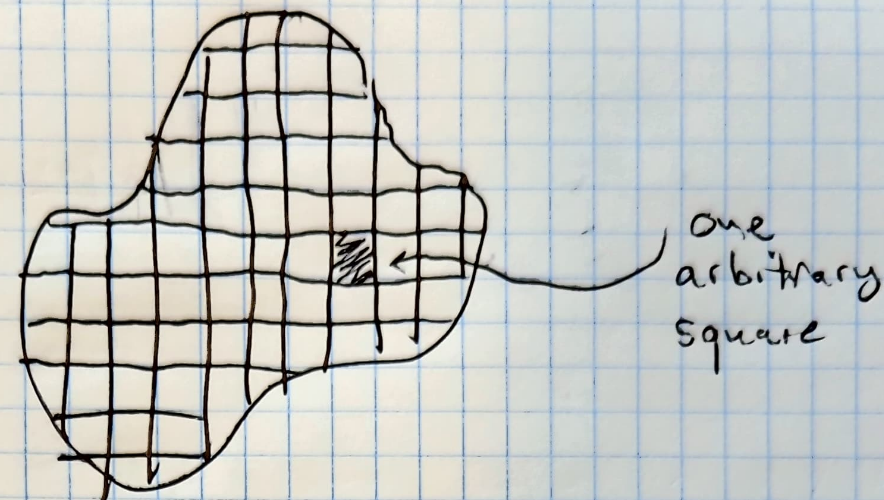
i.e.

Energy density is a function of (x,y)

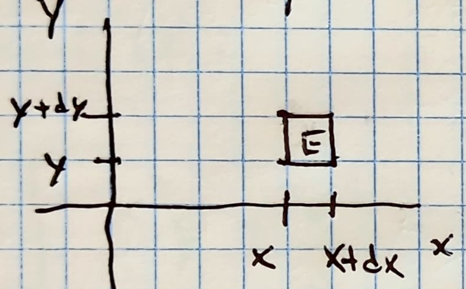
$E = E(x,y)$



The best way to do this is to break the region up into little squares, find the energy for each little square & add them up.



we ask "what is the area of our small square?"



geometry tells us $dA = dx\,dy$

the energy density of that square is $E(x,y)$ or $E(x+\frac{dx}{2}, y+\frac{dy}{2})$ etc (turns out $E(x,y)$ is just dandy)

So the total energy of our square is

$$E(x,y)\,dA = E(x,y)\,dx\,dy$$

Now we add them up & take $dx, dy \to 0$

$$E_{Total} = \int_{y_i}^{y_f}\int_{x_i}^{x_f} E(x,y)\,dx\,dy \quad \text{(for a rectangular region)}$$

E.G. $E(x,y) = xy + x^2 - y^3$

$$E_T = \int\int (xy + x^2 - y^3)\,dx\,dy$$

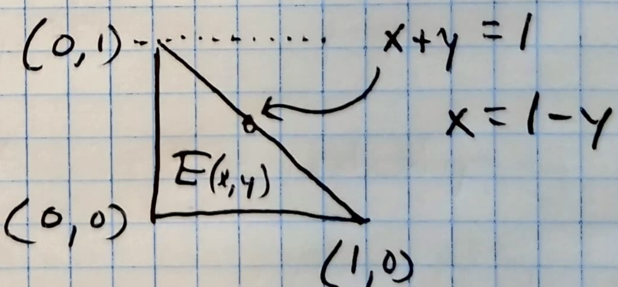
inside integral 1st:

$$= \int \frac{x^2}{2}y + \frac{x^3}{3} - y^3 x \; dy$$

$$= \frac{x^2y^2}{4} + \frac{x^3}{3}y - \frac{y^4 x}{4} + \text{constant}$$

If our region is not rectangular then our integration limits may be functions

E.G. a triangle



$$\int_0^1\int_0^{1-y} E(x,y)\,dx\,dy$$

in the x-integral $x = 1-y$

we can also find average values over a region. Just integrate & divide by the total area.

$$\text{Area} = \iint dA = \iint dx\,dy$$

e.g. area of a rectangle

$$A_R = \int_0^b \int_0^a dx\,dy = \int_0^b x\Big|_0^a dy$$

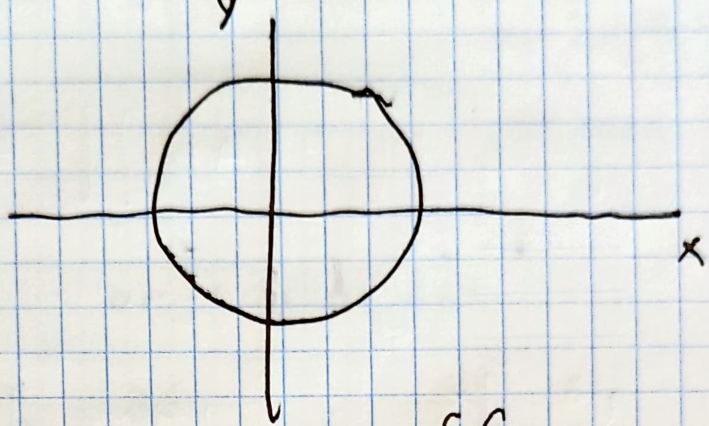
$$= \int_0^b a\,dy = \boxed{ab}$$

$$\langle E \rangle = \frac{\int E(x,y)\,dx\,dy}{\int dx\,dy}$$

$$\langle E^2 \rangle = \frac{\int E^2(x,y)\,dx\,dy}{\int dx\,dy}$$

etc

Often physical problems a rotationally symmetric & cartesian coordinates are cumbersome compared to polar suppose we have a circular region



we could again do this via $\iint dx\,dy$

but the integration limits are wierd thanks to

$$x^2 + y^2 = R^2$$

or

$$x^2 = R^2 - y^2 \longrightarrow \pm\sqrt{R^2 - y^2}$$

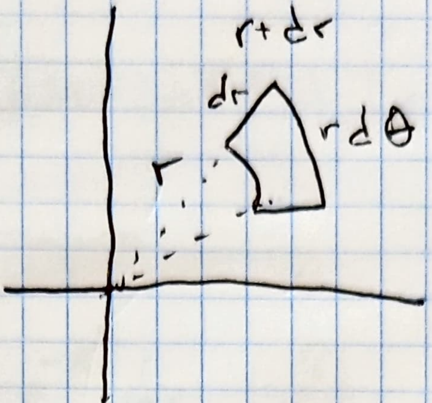
& it's a mess

but in ~~por~~ polar coordinates describing a circle is easy!

$$r = R$$

$$0 \leq \theta \leq 2\pi$$

so let's chop our region up using polar coordinates.



arc length $= r\,d\theta$

$\Rightarrow$ area $= r\,dr\,d\theta$

$$dA = r\,d\theta\,dr \quad \text{or} \quad r\,dr\,d\theta$$

the

$$A_{circle} = \int_0^{2\pi}\int_0^R r\,dr\,d\theta = 2\pi\frac{R^2}{2} = \pi R^2$$

you could also go a different route

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$dx = d(r\cos\theta) = dr\cos\theta + r\sin\theta\,d\theta$$
$$dy = d(r\sin\theta) = dr\sin\theta - r\cos\theta\,d\theta$$

then multiply $dx\,dy = r\,dr\,d\theta$

☆ (check this) ☆

☆ A MAJORLY IMPORTANT EXAMPLE ☆

How do we compute

$$I = \int e^{-x^2} dx \quad ?$$

we can't use a substitution or integrate by parts but we can change coordinates to polar

☆Trick☆ consider I^2

$$I^2 = \int e^{-x^2} dx \int e^{-y^2} dy$$

$$= \iint e^{-(x^2+y^2)} dx\, dy$$

A-HA! x^2+y^2 looks like a circle!
let's change to polar coords!

$$r^2 = x^2 + y^2$$

$$dx\,dy = r\,dr\,d\theta$$

initially we seem stuck.

we're trading e^{-x^2} for e^{-r^2} which looks the same. but the r in $r\,dr\,d\theta = dA$ saves the day!

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

notice the limits, when we switch to r we go from $0 \to \infty$; not $-\infty \to \infty$

the θ integral is easy $\int_0^{2\pi} d\theta = 2\pi$

so $I^2 = 2\pi \int e^{-r^2}\,r\,dr$

let $u = -r^2 \Rightarrow du = -2r\,dr$

$\Rightarrow \cancel{2r}\ r\,dr = -\frac{du}{2}$

$$\Rightarrow I^2 = \frac{2\pi}{2}\int_{-\infty}^{0} e^u\,du = \pi \ ! \text{ WOW!}$$

$$\Rightarrow \boxed{I = \sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\,dx}$$

★ HOMEWORK ★

solve $\tilde{I} = \int e^{-ax^2} dx$

you'll find $\tilde{I} = \sqrt{\frac{\pi}{a}}$

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once we have $\int e^{-ax^2} dx$

we can have BIG Fun

★for example★

$$\int x^2 e^{-ax^2} dx$$

doesn't $x^2 e^{-ax^2}$ look a lot like

$$-\frac{\partial}{\partial a} e^{-ax^2} ?$$

then $\int x^2 e^{-ax^2} dx = -\int \frac{\partial}{\partial a}(e^{-ax^2}) dx$

since we're not integrating over a we can pull $\frac{\partial}{\partial a}$ out of integral!

$$\Rightarrow \int x^2 e^{-ax^2}\,dx = -\frac{\partial}{\partial a}\int e^{-ax^2}\,dx$$

$$= -\frac{\partial}{\partial a}\left(\sqrt{\frac{\pi}{a}}\right) = -\sqrt{\pi}\left(\frac{\partial}{\partial a}a^{-\frac{1}{2}}\right)$$

$$= \frac{\sqrt{\pi}}{2}\,a^{-3/2}$$

$$\boxed{= \sqrt{\frac{\pi}{a}}\;\frac{1}{2a}}$$

recall that

$$\langle x\rangle = \int x\, f(x)\,dx$$ is the

average of x over some space.

$$\frac{1}{2a}\sqrt{\frac{\pi}{a}} = \langle x^2\rangle = \int x^2 f(x)\,dx$$

where $f(x)$ is e^{-ax^2}

this special $f(x) = e^{-ax^2}$ is whats known as the "Gaussian distribution" or "Bell Curve" or "Normal Distribution"

In its full glory

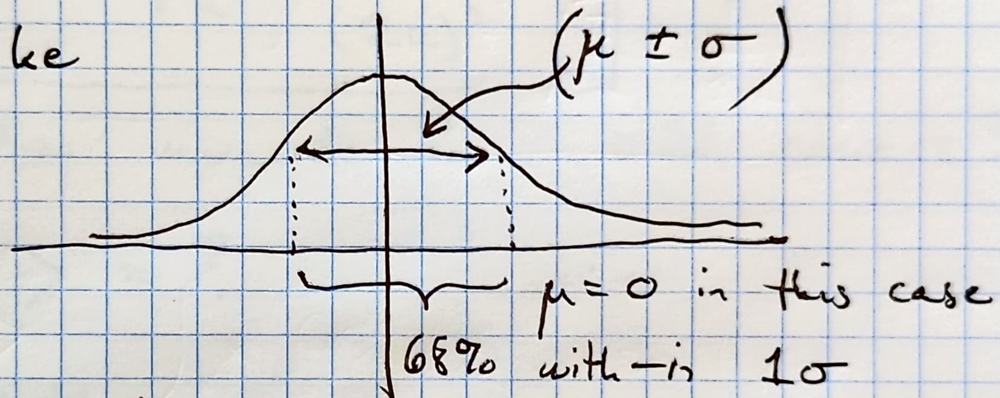
$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where

$\mu \equiv$ mean

$\sigma \equiv$ standard deviation

it looks like



&

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

HOMEWORK

Check the following

$$\mu = \langle x \rangle = \int_{-\infty}^{\infty} x\, p(x)\,dx$$

$$1 = \int_{-\infty}^{\infty} p(x)\,dx$$

$\sigma^2 \equiv$

we get a nice class of integrals with this trick called the "moments" of $p(x)$

$$\boxed{\langle x^{2n} \rangle = \int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \frac{(2n-1)!!}{(2a)^n}}$$

what are the moments for odd powers?

$$\langle x^{(2n+1)} \rangle = ?$$

★HOMEWORK★

Show $\boxed{\langle x^{2n+1} \rangle = 0}$ for Gaussian (take $\mu = 0$ & $\sigma = 1$)

★Hint★ use a trick you recently learned!

★ Think, then calculate ★

★PHYSICS★

Recall that for a box of gas the temperature is related to the average velocity of the molecules via

$$kT = \frac{1}{2} m \langle v_x^2 \rangle$$

in each dimension

In a 3D Box we have

$$kT = \frac{3}{2} m \langle v^2 \rangle$$

where $\langle v^2 \rangle = \langle v_x^2 + v_y^2 + v_z^2 \rangle = v_{rms}^2$

$v_{rms} = \sqrt{\langle v^2 \rangle}$

You might ask: "what probability function or distribution gives rise to this?"

Maxwell figured out this distribution of speeds is given by

$$\boxed{f(v_x) = \sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{m v_x^2}{2kT}}}$$

a Gaussian in v_x!

$f(v_x)$ tells us what fraction of molecules in our box have velocity v_x at some temperature T.

e.g. for small T we would expect the fraction of molecules moving crazy fast to be roughly zero

★ Check this ★ taylor expand $f(v_x)$ for $T \ll 1$ as a function of T

Now, because $f(v_x)$ represents a fraction of a total we demand

$$\int_{-\infty}^{\infty} f(v_x)\,dv_x = 1$$

This demand determines the $\sqrt{\frac{m}{2\pi kT}}$ factor in front.

Well its Gaussian so we can find the average SPEED. Empasis on "Speed" the average "velocity" will be ZERO due to symmetry [odd · even $\to$ 0]

$$\bar{s} = \langle |v_x| \rangle = \int_0^\infty 2\, v_x\, f(v_x)\, dx$$

Notice the odd/even trick doesn't work because $|v_x|$ is even & $f(v_x)$ is even

$$\langle v_x \rangle = 0$$

but $\langle |v_x| \rangle \neq 0$

you can do this integral (TRY) & find that

$$\boxed{\bar{s} = \langle |v_x| \rangle = \sqrt{\frac{2kT}{\pi m}}}$$

This has all been in 1D

for 3D we have

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

&

$$\boxed{f(v) = 4\pi^2 v^2 \left[\frac{m}{2\pi kT}\right]^{3/2} e^{-\frac{mv^2}{2kT}}}$$

Notice the v^2 term in front now.

this term comes from "spherical coordinates"

$$\therefore \int dx\,dy\,dz \longrightarrow \int r^2 \sin\theta \, dr\, d\theta\, d\phi$$

similar to changing from cartesian to polar

Polar

$x = r\cos\theta$ $\quad dA = dx\,dy = r\,dr\,d\theta$

$y = r\sin\theta$

SPHERICAL

$x = \cancel{r\cos\theta\sin\phi}\ r\cos\theta\sin\phi$

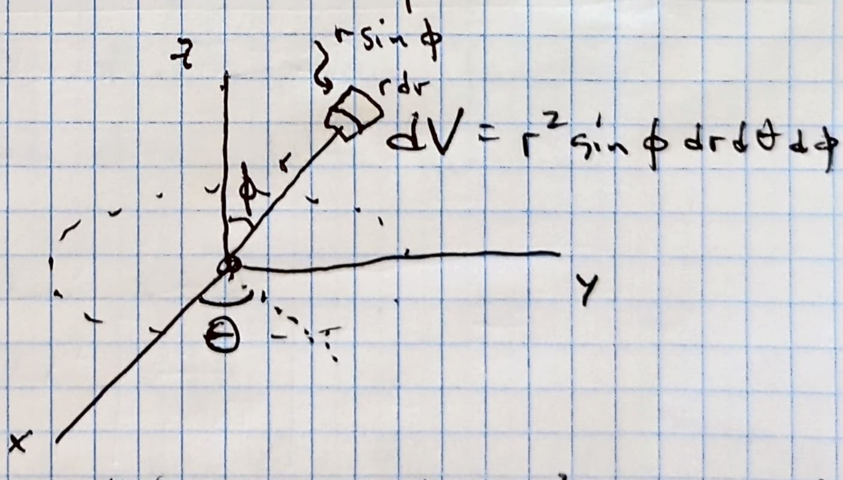
$y = r\sin\theta\sin\phi$

$z = r\cos\phi$

$$dV = dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

ϕ: azimuthal angle

θ: polar angle

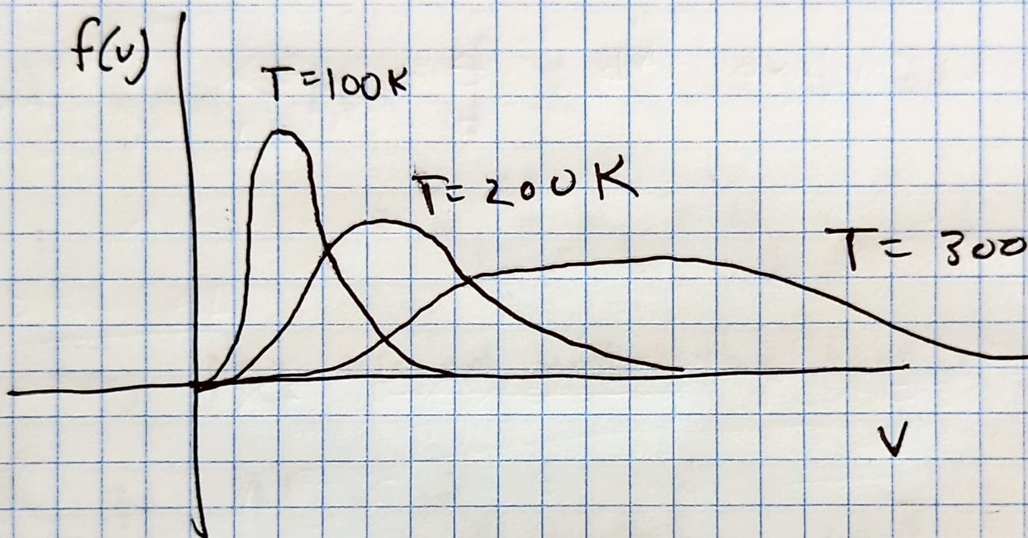


Someday you'll memorize this; especially $r^2\sin\theta\,dr\,d\theta\,d\phi$

$$V_{sp} = \frac{4}{3}\pi R^3 = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

you won't need to use spherical coordinates yet but that is why $f(v_x)$ ≑ $\tilde{f}(v)$ have different forms

$f(v)$ looks like



as T grows the standard deviation increases
as $T \to 0$ we expect $f(v) = 1$ for $v=0$
≑ $f(v)=0$ for $v \neq 0$ this specific "function" has a name "the dirac-delta-function"

$$\delta(x) = 1 \quad \text{when } x=0$$
$$= 0 \quad \text{when } x \neq 0$$

$\delta(x)$ allows us to do useful things like
if you have a bunch of point masses
we can write their density as $\rho(x) = \sum m_i \delta(x - x_i)$

usually we integrate $\delta(x)$ ~~a ge~~ against a function

$$\int \delta(x-a)\, f(x)\, dx = f(a)$$

which ends up being SUPER Handy.

$$\text{total mass} = M = \int \sum \delta(x-x_i)\, m_i \, dx$$

$$= m_1 + m_2 + m_3 + \dots,$$

we may play around with this further later.

Back to Maxwell,

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}}$$

find $\langle |v| \rangle$ = average speed

$$\langle |v| \rangle = 2\int_0^\infty v\, f(v)\, dv$$

using our Gaussian integral from earlier.