Fun with Gamma, Beta, & error functions

Gamma

The Γ function is defined by integrals of the type

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \qquad (n > 0)$$

which, as you can imagine, appear A LOT!
This can't be evaluated analytically, hence we just call it $\Gamma(n)$

let's evaluate $\Gamma(n+1)$

$$\Gamma(n+1) = \int_0^\infty x^{(n+1)-1} e^{-x}\, dx = \int_0^\infty x^n e^{-x}\, dx$$

you should be very tempted to integrate by parts

$$\Gamma(n+1) = \left[-x^n e^{-x} \right]_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\, dx$$

verify that $\boxed{\Gamma(n+1) = n\Gamma(n)}$ hint e^x always beats x^n as $x \to \infty$

notice that $\Gamma(1) = 1$

$$\Rightarrow \Gamma(2) = 1 = \Gamma(1+1)$$

$$\Rightarrow \Gamma(2+1) = 2\Gamma(2)$$

$$\Rightarrow \Gamma(3+1) = 3\Gamma(3) = 3 \times 2\Gamma(2)$$

$$= 3 \times 2 \times 1 = 3!$$

by induction $\boxed{\Gamma(n+1) = n!}$

$\Gamma(n+1)$ gives us a new definition of factorial even for - values of n

let $x = y^2$ then $\Gamma(n) = 2\int_0^\infty y^{2n-1} e^{-y^2}\, dy$

$dx = 2y\,dy$

let $n = \frac{1}{2} \Rightarrow \Gamma(\frac{1}{2}) = 2\int_0^\infty y^0 e^{-y^2}\, dy$

but thats a gaussian

$$\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$$

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

from $\Gamma(n+1) = n\Gamma(n)$ you get all sorts of values!

Here's one last crazy useful idea,

put the x^n into an exponential by $e^{n \ln x} = x^n$

then $$n! = \int_0^\infty e^{n\ln x - x}\, dx$$

now change variables $x = n + y$ or $y = x - n$

this isn't an obvious thing to do but

$dx = dy$ &

$$\ln x = \ln n + \ln\left(1 + y/n\right)$$

<u>for large n</u> we can taylor expand $\ln\left(1 + y/n\right)$

$$\ln\left(1 + \frac{y}{n}\right) \approx \frac{y}{n} - \frac{y^2}{2n^2} + \ldots$$

plugging back into $\Gamma(n)$

$$n! = \int_{-n}^{\infty} e^{n\left[\ln n + y/n - \frac{y^2}{2n^2} + \ldots\right] - n - y}\, dy$$

$$\approx e^{n\ln n - n} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2n}}\, dy = e^{n\ln n - n}\sqrt{2\pi n}$$

because we let n become very large

$\therefore$ $$\boxed{n! \approx n^n e^{-n}\sqrt{2\pi n}}$$

This is called Stirling's approximation & is VERY useful!

Beta

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^\infty \frac{y^{n-1}\,dy}{(1+y)^{m+n}}$$

$B(m,n)$ isn't terribly useful but it is related to $\Gamma(n)$ by

$$\boxed{B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

this form actually shows up in String Theory!

Error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,du$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\,du$$

This guy shows up in probability. one generally looks up ~~its~~ its values or numerically compute them

$\Gamma(n)$ & Spheres

we saw $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ & prompts us to wonder "where the heck are circles?!"

lets find the volume of a sphere.

consider 3D (x, y, z)

a sphere is the surface defined by

$$x^2 + y^2 + z^2 = R^2$$

when we say "sphere" we always mean the 2D surface.

the "Ball" is the surface along with the interior.

The volume of a Ball is given by

$$V_S = \int_S dV = \int r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{4}{3}\pi R^3$$

what about arbitrary spheres of any dimension?

again we want to solve

$$V_{HS} = \int dV_n \quad \text{where}$$

$$x_1^2 + x_2^2 + \dots x_n^2 = R^2$$

is an n-dimensional Sphere (the surface is (n-1) dimensional) if we knew dV_n we could just find V_{HS}

1D	2D	3D
$dV_1 = dr$	$dV_2 = r\,dr\,d\theta$	$dV_3 = r^2 \sin\theta\, dr\, d\theta\, d\phi$

the angular parts of dV is often denoted $d\Omega$ where Ω is a solid angle

eg $\quad dV_3 = r^2\, dr\, d\Omega_2$

Now, notice what $\int d\Omega_2$ gives us.

$$\int d\Omega = \text{Surface area of our sphere}$$

in arbitrary dimensions we expect

$$\boxed{dV_n = r^{n-1}\, dr\, d\Omega_{n-1}}$$

Now, if we could somehow figure out the surface area of a sphere then we could easily find the volume by integrating

$$\int r^{n-1}\, dr \quad \text{\& multiplying by } S_{n-1} = \int d\Omega_{n-1}$$

Here's the insanely cool trick!

compute $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = I$

we previously found this by squaring then going to polar coords & solving

$$I^2 = \iint e^{-x^2-y^2}\,dx\,dy$$

$$\Rightarrow \int_0^{2\pi}\int_0^{\infty} r\,dr\,\underbrace{d\theta}_{d\Omega}\, e^{-r^2} = \pi$$

$$\Rightarrow I = \sqrt{\pi}$$

we're more than welcome to compute I^3

$$I^3 = \iiint e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz$$

$$= (\sqrt{\pi})^3$$

or even $I^n = (\sqrt{\pi})^n = \displaystyle\int e^{-(x_1^2+x_2^2+\dots x_n^2)}\,dx_1\,dx_2\,dx_3\cdots dx_n$

But in polar coordinates we get that same angular part $d\Omega$

$$I^n = \pi^{n/2} = \int r^{n-1} e^{-r^2}\,dr\,d\Omega$$

Oh my god look at that radial integral!

does it look similar to something you've seen?!

$$\Gamma(n) = 2\int_0^\infty y^{2n-1} e^{-y^2}\, dy$$

so that our radial integral is "simply"

$$\frac{1}{2}\Gamma\left(\frac{n}{2}\right)$$

all ~~to get~~ together

$$I^n = \pi^{n/2} = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)\int d\Omega$$

$$= \frac{1}{2}\Gamma\left(\frac{n}{2}\right)(\text{Surface area})$$

solving for the surface area

$$\boxed{S_{n-1} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}}$$

then the $\boxed{\text{Volume} = \int_0^R S_n r^{n-1}\, dr = \frac{2R^n \pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)}}$

is the volume of an n-dimensional sphere. Wow!