Introduction to Diff EQs

The most important equations in all of science are differential equations. These is a new mathematical object VERY similar to functions that appear everywhere. It's called an "operator" and these "operators" are at the heart of Quantum Theory.

Recall the definition of a function.

a function maps a set of numbers (or even ANY set of objects, e.g. apples) into another set of numbers

abstractly we write

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

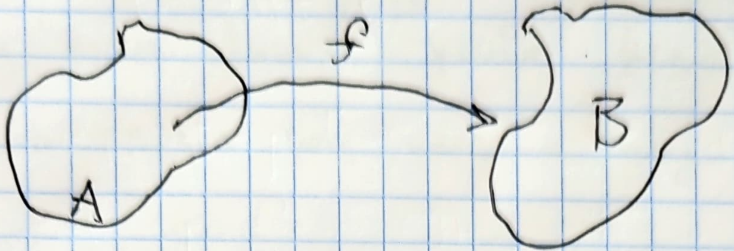
where $\mathbb{R}$ is the set of real numbers.

e.g. $f(x) = x^2$ maps

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ \end{pmatrix} \begin{matrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \\ \longrightarrow \end{matrix} \begin{pmatrix} 1 \\ 4 \\ 9 \\ \\ \end{pmatrix}$$

et cetera

we sometimes draw it out



or $A \xrightarrow{f} B$

we can do composition of functions

e.g. $f : A \to B$

& $g : B \to C$

then $g \circ f : A \to C$ $\quad$ $g \circ f$ we read as "g after f"

diagramatically

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

inverse functions go the other way.

e.g. if f^{-1} exists then

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

& $f^{-1} \circ f : A \to A$

is the identity function

you can do A <u>LOT</u> with <u>objects</u> and <u>arrows</u>!

it's a very abstract branch of math called "Category Theory" sometimes referred to as the "primordial ooze" of Math or even the Math of Math

Anyhow, perhaps you've never thought of functions in this way even though you've been using special functions since you were a baby.

addition: adding is a special function that takes two numbers & spits out one number

we write

$$+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

$\mathbb{R} \times \mathbb{R}$ is the cartesian product of sets. it's just a pair of numbers (x, y)

e.g. $+(3, 7) = 10$

$+$ can also be viewed tentatively as a function that takes a vector & outputs a number.

we can have vector valued functions like

$$f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

f takes one vector & outputs another vector.

By golly that sounds like a matrix!

$$A\vec{x} = \vec{y}$$

we could go ong on this path but we need "operators"

An operator is an object that maps functions into other functions

$$O : f \to g$$

$\frac{d}{dx}$ is an operator

$$\frac{d}{dx} f = \frac{df}{dx}$$

if $\cancel{*}$ $f(x) = x^2$ then $g(x) = 2x$

$$f \xrightarrow{\frac{d}{dx}} g$$

we can have all sorts of operators

the operator $\cancel{D = \frac{d^2}{dx^2}}$ $\cancel{*}$

$$\boxed{D = \frac{d^2}{dt^2} + \omega^2}$$

is an operator that can describe a harmonic oscillator

e.g $\Sigma F = m\ddot{x}$ $\cancel{mkx}$ $= -kx$

$$\therefore \; m\frac{d^2x}{dt^2} + kx = 0$$

We would LOVE to know the position of the mass at all points in time.

That is, we must solve the diff EQ

$$\boxed{Dx = 0 = \frac{d^2x}{dt^2} + \omega^2 x = 0} \quad (1)$$

Diff EQ's are generally VERY hard to solve. Abstractly we have

$$x(t) \xrightarrow{D} 0$$

Often, to solve these equations, we make a seasonable guess or "ansatz" based on our intuition.

for this system we expect sine or cosine solutions. It's a mass on a spring! It bounces up and down! So we guess

$$x(t) = A\cos(\omega t)$$

a more sophisticated person would invoke euler & write

$$x(t) = Ae^{i\omega t}$$

because $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

then take the Real part of $x(t)$ at the end!

let's plug in our ansatz into (1)

$$Dx = 0$$

$$D\left[Ae^{i\omega t}\right] = 0$$

$$\left[\frac{d^2}{dt^2} + \omega^2\right] Ae^{i\omega t} = 0$$

$$\ddot{x} = A(i\omega)^2 e^{i\omega t}$$

So our equation is

$$-A\omega^2 e^{i\omega t} + A\omega^2 e^{i\omega t} = 0$$

By golly our guess works!

So $x(t) = Ae^{i\omega t}$ is THE solution

But what is our amplitude A ?

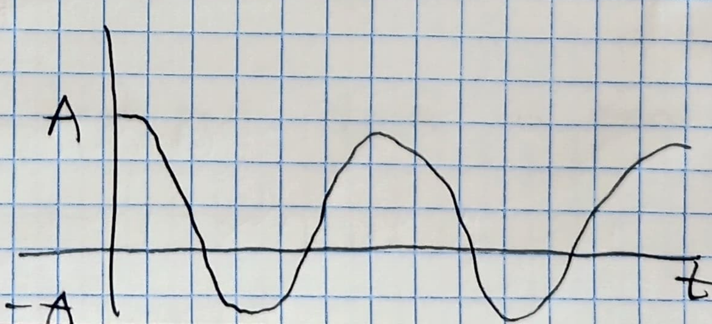
Be clever & note

$$x(t=0) = A$$

A is literally where the mass is at $t=0$

if we stretch it 10meters & let go A will be 10

our x(t) looks like



Now, don't get me wrong, there are very many strict & rigorous ways of solving diff EQs and they can be very long and messy. But physicists are practical minded people & can usually by-pass all that by clever guesses.

Now, In General, you MUST have a few things handy when solving diff eq's

① initial data.

if you have a $\frac{d^2}{dt^2}$ type diff EQ you MUST know, or be given both the initial position AND the initial velocity. if it were of $\frac{d^3}{dt^3}$ you would ALSO require the initial acceleration.

for $\frac{d^n}{dt^n}$ you need n initial conditions

so, $x(0)$, $\dot{x}(0)$, $\ddot{x}(0)$, ... etc

In our SHO $x(0) = A$ & $\dot{x}(0) = iA\omega$

but $\dot{x}$ is complex, the real part of $\dot{x}$ is zero with just these ideas you can solve a LOT of diff EQ's

Let's solve the "damped harmonic oscillator"

our damping force will be due to wind resistance which, if you recall, is something like

$$F_{Damp} = -2\gamma V = -2\gamma \frac{dx}{dt}$$

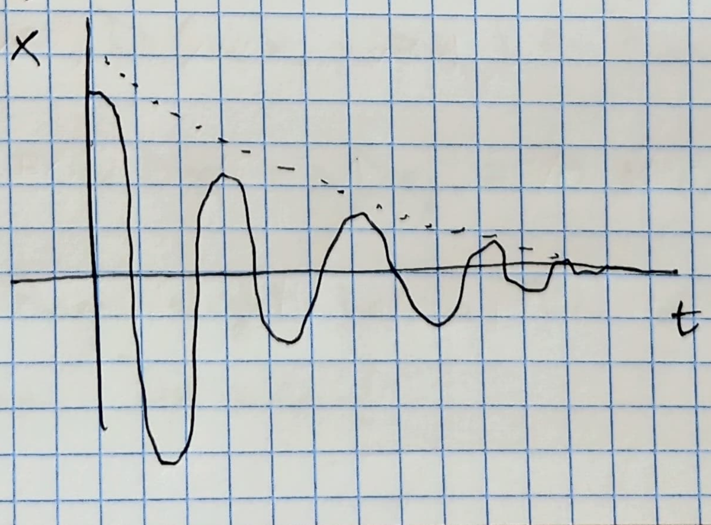
So, for our mass we have

$$Dx = 0 \quad \text{where NOW} \quad \boxed{D = \frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2}$$

the $\frac{d}{dt}$ term makes our previous solution invalid. What do we expect $x(t)$ should be?

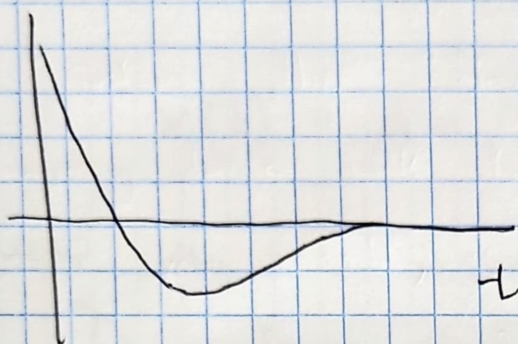
well it depends on how big γ is

you might guess something like

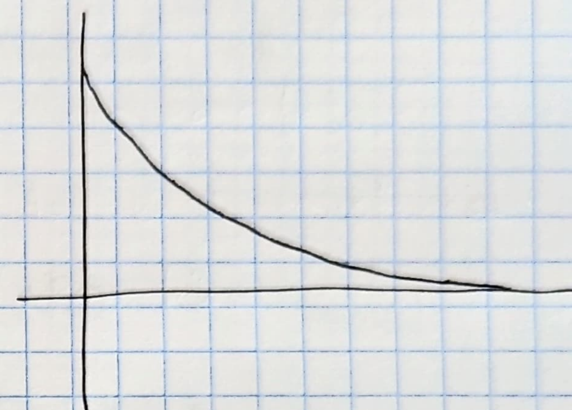


a wave that is decaying exponentially.

or we might have



or



Many different functions! How to choose?!!

a not so obvious observation

$Dx = 0$ is Linear.

if $x_1(t)$ & $x_2(t)$ <u>both</u> solve $Dx = 0$

then, by linearity

$x(t) = \alpha x_1(t) + \beta x_2(t)$ also solves it

why?

$Dx_1 = 0$ & $Dx_2 = 0$

$\Rightarrow Dx = D(\alpha x_1 + \beta x_2) = D\alpha x_1 + D\beta x_2$

$= \alpha D x_1 + \beta D x_2 = 0 + 0$

QED.

so, infact, there might be a TON of functions that solve $Dx = 0$!!

sit back & watch the magic of complex numbers.

we choose a candidate solution of

$$x(t) = Ae^{-i\omega t}$$

where NOW the ω in $x(t)$ is NOT the same as in D. Does this function work?

plug it into $Dx = 0$

$$\dot{x} = \frac{dx}{dt} = -Ai\omega e^{-i\omega t}$$

$$\ddot{x} = \frac{d^2x}{dt^2} = A(i\omega)^2 e^{-i\omega t}$$

$$\Rightarrow Dx = -\omega^2 Ae^{-i\omega t} - 2Ai\gamma\omega e^{-i\omega t} + A\omega_0^2 e^{-i\omega t} = 0$$

Magically all the $Ae^{-i\omega t}$ divide out & so

$$\boxed{-\omega^2 - 2i\gamma\omega + \omega_0^2 = 0}$$

well you can solve this easy enough. Remember we're trying to figure out if there is some ω that will allow $e^{-i\omega t}$ to work as a solution.

So,

$$\boxed{\omega = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}}$$

This tells us there are TWO different ω that solves $Dx = 0$

ω_+ & ω_-

our general solution is a linear combination of these

$$x(t) = Ae^{-i\omega_+ t} + Be^{-i\omega_- t}$$

$$= Ae^{-i(-i\gamma + \sqrt{\omega_0^2 - \gamma^2})t}$$
$$+ Be^{-i(-i\gamma - \sqrt{\omega_0^2 - \gamma^2})t}$$

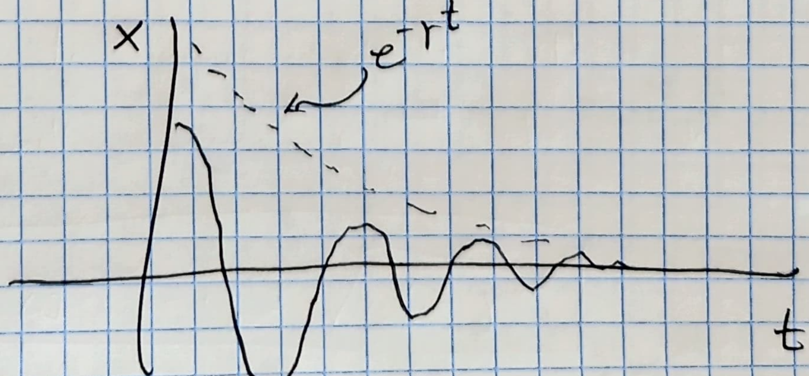
Look at this MAGIC! $-i(-i\gamma) = -\gamma$!

$$\boxed{x(t) = Ae^{-\gamma t} e^{-i\sqrt{\omega_0^2 - \gamma^2}\, t} + Be^{-\gamma t} e^{-i\sqrt{\omega_0^2 - \gamma^2}\, t}}$$

$e^{-\gamma t}$ is an exponential decay!

& $e^{-i\sqrt{\omega_0^2 - \gamma^2}\, t}$ is just a wave!

the product is



A & B can be determined from initial conditions.
they are two unknowns but
$x(0)$ & $\dot{x}(0)$ gives us two equations

A & B might also be complex numbers

the final task is to find $Re[x(t)]$
& compare to experiment

Trig IDs & complex numbers lead to

$$Re[x(t)] = Ce^{-\gamma t}\cos(\Omega t + \phi)$$

$$\Omega = \sqrt{\omega_0^2 - \gamma^2}$$

and $\phi = \tan^{-1} \frac{\gamma\omega}{\omega_0^2 - \omega^2}$

you can check this using the following exercise

<u>exercise</u> $z = (a+bi)e^{i\gamma}$

write z in polar form

$$(a+bi) = \sqrt{a^2+b^2}\, e^{i\phi} \qquad \phi = \tan^{-1}\frac{b}{a}$$

then $\boxed{z = \sqrt{a^2+b^2}\, e^{i(\phi+\gamma)}}$

Same idea for $x(t) = e^{-\gamma t}\left[Ae^{i\Omega t} + Be^{-i\Omega t}\right]$