Logs and Logs and logs

☆ Given $a^y = x$ we can define a new function called the logarithm by the notation

$$\log_a x = y$$

where a is called the "base" and y is the exponent.

Usually we write $\log x = y$ if we are using base 10

We also use $\ln x = y$ called the "natural log" if our base is $e \approx 2.718281828...$

in computer science base 2 is common.

eg $$2^8 = 256$$

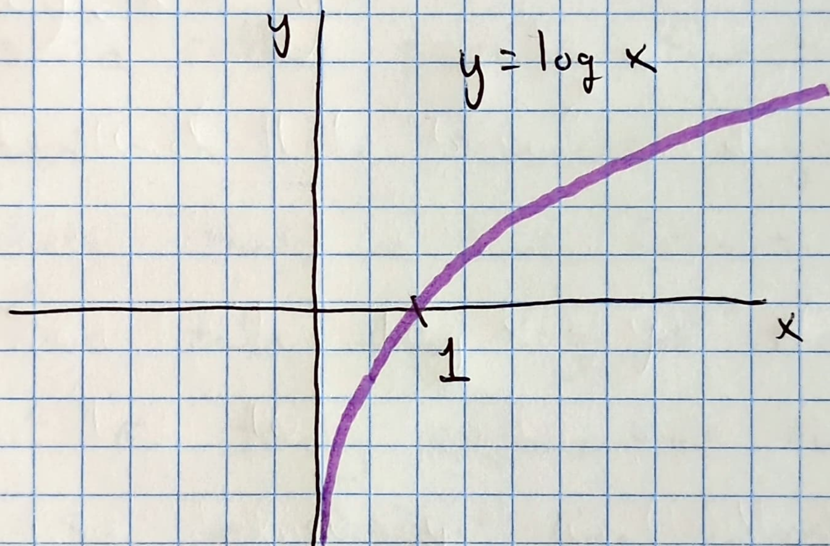
$$\log_2(256) = 8$$

The log gives us the exponent we need to raise the base to in order to get the input number.

Logs are great for handling really large numbers. for example;

$\log 10{,}000{,}000 = 7$ (base 10)

Because of this the log function grows VERY VERY slowly. It looks like



some special values are

$\log 0$ = Not defined $\longrightarrow -\infty$

$\log 1 = 0$

let's see why $\log 0$ is not defined.

log says to take 10 and raise it to y and get x

$\therefore \log_{10} 0 = 10^y = 0$

this asks for the exponent that gives 0

But there is no number.

we notice that $\frac{1}{10^2} = 0.01$ and

as we make the exponent more negative the number gets closer to zero.

viz $10^{-7} = 0.0000001$ etc.

So log is a rather peculiar function. It is used in the sciences any time we have data that we think follows an exponential growth. if we take the log of that data then if it is truly exponential then the result will be a straight line whose slope is the exponent.

Eg. we get some data (say growth over time)

~~time height 0 5 1 50 2~~

time	height
0	5
1	50
2	500
3	5000
.	
.	
.	

we plot this data and find it follows

h = height $= 5\,10^t$

Now we take the log h (base 10) and this gives

$$\log(5\,10^t) = \log 5 + t \log 10$$
$$= \log 5 + t$$
$$= 0.7 + t \quad \sim y = mx + b$$

this is the form of a line with ~~exponent~~ slope 1 and sure enough our exponential was 10^t

This is the general idea of logs but I havent shown the rules.

Algebra of logs

$$a^y = x \Rightarrow \log_a x = y$$

$$\log_a(\alpha \cdot \beta) = \log_a(\alpha) + \log_a(\beta)$$

$$\log_a\left(\frac{\alpha}{\beta}\right) = \log_a(\alpha) - \log_a(\beta)$$

$$\log_a(\alpha^k) = k \log_a(\alpha)$$

$$\log_a(a^k) = k \log_a(a) = k$$

$$\log_a(a) = 1$$

$$a^{\log_a(k)} = k$$

using these rules allow us to solve complicated equations.

$\boxed{\text{Ex 1}}$ In the following equation our strategy will be to "bring down" the exponents using the log function. It is similar to how we would ~~you~~ use arctan or $\tan^{-1}$ in order to solve for an angle $\phi = \tan^{-1}\frac{y}{x}$.

Suppose $(2^3)^{x+1} = 5 \cdot 2^{7x-5}$ SCARY!

$\boxed{\text{Step 1}}$: $\star\ (a^b)^\alpha = a^{\alpha b}\ \star$ use this on $(2^3)^{x+1}$

you might be tempted to write 8^{x+1}

However, we have a base 2 on the right side and we will want base 2 on the left. This is the generall ~~sta~~ strategy. Match bases on both sides

We have

$$2^{3x+3} = 5\, 2^{7x-5}$$

$\boxed{\text{Step 2}}$: apply $\log_2$ to both sides. We say:

$\star$ "bring down the exponents" $\star$

$$\log_2 2^{3x+3} = \log_2(5 \cdot 2^{7x-5})$$

$\boxed{\text{Step 3}}$: we will apply the algebraic rules of log to each side

* Left side *

$$\log_2(2^{3x+3}) = 3x+3\log_2(2)$$

but $\log_2(2) = 1$ so the left side reduces to $\boxed{3x+3}$

* Right Side *

$$\log_2(5 \cdot 2^{7x-5}) = \log_2(5) + \log_2(2^{(7x-5)})$$

using the product rule of log

now, $\log_2 5$ is just a number. it is the number you need to raise 2 by to get 5

it is 2.23 or so, you can use a calculator

we have $\quad 2.23 + \log_2(2^{7x-5})$

$$= 2.23 + (7x-5)\log_2 2$$

$$\boxed{= 2.23 + 7x - 5}$$

this is our right side.

$\boxed{\text{Step 4}}$: solve for x

$$\boxed{3x + 3 = 7x - 5 + 2.23 \ldots}$$

The log function initially looks scary but always remember that it is "sort of" like an inverse trig function but for exponentials. with practice it will be come second nature.

In general if you see a bunch of exponents then you think "Log Log Log!!"

Let's try another!

$\boxed{\text{Ex 2}}$

$$\frac{\log(16-x^2)}{\log(3x-4)} = 2$$

Solve for x.

This seems backwards from our previous example. all the x's are on the same side but the logs confound everything.

Let us try to use the algebraic property of log & see if we notice anything.

$\boxed{\text{Step 1}}$ Get rid of fraction

$$\log(16-x^2) = 2\log(3x-4)$$

Now we notice that if we get the logs the same then maybe we can solve!

$\boxed{\text{Step 2}}$ use exponent rule of log on the RHS

$$\log(\alpha^{\beta}) = \beta \log \alpha$$

$$2\log(3x-4) = \log(3x-4)^2$$

$$\therefore \quad \log(16-x^2) = \log(3x-4)^2$$

Please remember that the right hand side does NOT say to square the log.

i.e.

$$\left[\log(3x-4)\right]^2 \neq \log(3x-4)^2$$

Now we have found $\log(a) = \log(b)$ then certainly $a = b$

so we solve

$$16 - x^2 = (3x-4)^2$$

$$\Rightarrow \boxed{x = 0, \ 12/5}$$

All of these kinds of problems require remembering the algebraic rules. The more problems you complete the easier you will memorize them.

Solve: $$\frac{10}{1+e^{-x}} = 2$$

This kind of equation is VERY common in quantum mechanics. x represents the energy of something but that's not important.

$\boxed{\text{Step 1}}$ Get rid of fraction

$$10 = 2(1+e^{-x})$$

$\boxed{\text{Step 2}}$ take logs.

In this case we MUST use the "natural log" $\ln = \log_e$

It follows the same rules and is just a very important base called e

e is just a number: $e = 2.71828\ldots$

You find out WHY e an ln in calculus!

$$\therefore \quad \log_e(10) = \ln(10) = \log_e\left[2(1+e^{-x})\right]$$
$$= \ln\left[2(1+e^{-x})\right]$$

$\ln 10 = 2.30\ldots$

$$2.30\ldots = \ln\left[2(1+e^{-x})\right] = \ln 2 + \ln(1+e^{-x})$$

$\ln 2 = 2.48\ldots$

we have

$$2.30 = 2.48 + \ln(1+e^{-x})$$

or $\ln 10 = \ln 2 + \ln(1+e^{-x})$

move $\ln 2$ over

$$\ln 10 - \ln 2 = \ln(1+e^{-x})$$

$= \ln \frac{10}{2} = \ln(1+e^{-x})$ using quotient rule

$= \ln 5 = \ln(1+e^{-x})$

$\therefore$ $5 = 1 + e^{-x}$ using $\ln a = \ln b \Rightarrow a = b$

then $4 = e^{-x}$

Now we apply the natural log AGAIN!

$\ln 4 = \ln e^{-x} = -x \ln e$ (exponent rule)

$-\ln 4 = x$ because $\ln e = 1 = \log_e e$

Notice in this problem I used ln twice.

I didnt need to though.

Look, we had $10 = 2(1+e^{-x}) = 2 + 2e^{-x}$

then I can move things around or divide by 2

$$\frac{10}{2} = 1 + e^{-x} = 5$$

$$\Rightarrow 4 = e^{-x}$$

and HERE I can apply ln to get

$$x = -\ln 4.$$

The point is, is that you can use log as much as you want to solve a problem so long as you use the correct algebra.

☆ WARNING ☆

$$\log(1+x) \neq \log(1) + \log(x)$$

this is just like how $\sin(\alpha+\beta) \neq \sin\alpha + \sin\beta$.

Be careful!

Prove

$$\log_{1/\sqrt{6}} \sqrt{x} = -\log_6 x$$

YIKES!

We need a new formula! this formula allows us to change bases and is VERY

☆ Important ☆

$$\boxed{\log_x(y) = \frac{\log_w(y)}{\log_w(x)}}$$

I never ~~[illegible]~~ remember this formula!
Here W can be anything you want.
w = 2 or 10 or even e are great choices
to use. We will use w = 10

$$\log_{1/\sqrt{b}} \sqrt{x} = \frac{\log_{10} \sqrt{x}}{\log_{10} 1/\sqrt{b}} = \frac{\log \sqrt{x}}{\log 1/\sqrt{b}}$$

let's look at only the denominator for a second
$\log 1/\sqrt{b}$. the $1/\sqrt{b}$ can be written
as $(b)^{-1/2}$ then we can use the
exponent rule.

☆ PRO TIP ☆ In general write <u>all</u>
roots as exponents. $\sqrt{}$ is just extra
and pointless notation.
eg $\sqrt[3]{y} = y^{1/3}$. <u>always</u> write numbers!

okay, rewriting everything with exponents

we have

$$\frac{\log(x^{1/2})}{\log(b^{-1/2})}$$

then we use the exponent rule of log to "bring down" the exponents

$$\frac{\log(x^{1/2})}{\log(b^{-1/2})} = \frac{\cancel{\tfrac{1}{2}}\log x}{-\cancel{\tfrac{1}{2}}\log b} = -\frac{\log x}{\log b}$$

But this looks JUST LIKE our base change formula from when we started! we are trying to prove $\log_{1/\sqrt{b}} \sqrt{x} = -\log_b x$ so we had better go back to base b.

recalling our base change formula

$$\log_{\underbrace{x}_{\text{base}}}(\overbrace{y}^{\text{value}}) = \frac{\log_w(y)}{\log_w(x)}$$

(Not the same x as above)

we have $\boxed{-\frac{\log x}{\log b} = -\log_b x}$ QED